INDIAN STATISTICAL INSTITUTE

Rings: Assignment I

MMath 1st year

Algebra I

1. Let R be a ring with 1. Prove that
   (a) $a.0 = 0.a = 0$
   (b) $(-a).b = -ab$ and $a(-b) = -ab$
   (c) if $ab = ba$, then $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + b^n$.
2. Let R be a ring with 1. Prove that if all the axioms for a ring *except* the commutativity of addition are assumed, then commutativity of addition follows.
3. Let R be a ring with 1 and let $a, b \in R$. If $(1 - ab)$ is invertible, then show that $(1 - ba)$ is also invertible.
4. Show that in $C([0,1])$, an element f is a zero-divisor if and only if the set of points where f vanishes contains an open interval.
5. Let I be an ideal of a ring R (with $1 \neq 0$). Consider the set $U = \{a \in R^* \mid a \equiv 1 \bmod I\} \subset R^*$. Show that U is a normal subgroup of R^*.
6. Let R be a commutative ring with $1 \neq 0$. Prove that every ideal of $M_n(R)$ is of the form $M_n(I)$ for some ideal I of R and that $I \mapsto M_n(I)$ is a bijective map from the set of ideals of R to the set of ideals of $M_n(R)$.
7. Let R be a commutative ring with $1 \neq 0$. Prove that $M_n(R[X]) \simeq M_n(R)[X]$.
8. Let R be a commutative ring with $1 \neq 0$. An element $a \in R$ is called *nilpotent* if there is $k > 0$ such that $a^k = 0$. Prove the following.
   (a) If $u \in R^*$ and a is nilpotent, then $u + a \in R^*$ (in particular, $1 + a \in R^*$).
   (b) If a, b are nilpotent, then so are $a \pm b$.
   (c) Let $\mathfrak{n} = \{a \in R \mid a \text{ is nilpotent}\}$. Prove that $\mathfrak{n}$ is an ideal of R.
   (d) Prove that $R/\mathfrak{n}$ does not have any non-zero nilpotent element.
9. Find all the units of the ring $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$.
10. Let $\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$. Is this ring isomorphic to $\mathbb{Z}[\sqrt{-5}]$?
11. Let G be an abelian group and $R = \{\phi : G \to G \mid \phi \text{ is a morphism of groups}\}$. Define addition on R by $(\phi + \psi)(g) = \phi(g) + \psi(g)$ and multiplication by $(\phi\psi)(g) = \phi(\psi(g))$. Show that R is a ring with identity but is not commutative in general.
12. Let R be a commutative ring with $1 \neq 0$. Prove that the product of two principal ideals is principal.
13. Let R be a commutative ring with $1 \neq 0$. Consider the ideal $I =< a, b >$, where $a, b \in R$. Let $\lambda \in R$. Consider the ideal $J =< a + \lambda b, b >$. Prove that $I = J$.
14. In the ring $\mathbb{Z}[X]$ prove that $< 2 > \cap < X >=< 2X >$.
15. Prove that the ring $\frac{\mathbb{Z}[X]}{<X^2+5>}$ is isomorphic to $\mathbb{Z}[\sqrt{-5}]$.
16. Let R be a commutative ring with $1 \neq 0$. Let I be an ideal of R and $\text{rad}(I) := \{a \in R \mid a^n \in I \text{ for some } n \geq 1\}$. Prove the following:
    (a) $\text{rad}(\text{rad}(I)) = \text{rad}(I)$

(b) $\text{rad}(IJ) = \text{rad}(I \cap J) = \text{rad}(I) \cap \text{rad}(J)$

(c) $\text{rad}(I) = R$ if and only if $I = R$

(d) $\text{rad}(I + J) = \text{rad}(\text{rad}(I) + \text{rad}(J))$

(e) If $P \in \text{Spec}(R)$, then $\text{rad}(P^n) = P$ for all $n > 0$

17. Let R be a commutative ring with $1 \neq 0$. Prove that the nilradical of $R[X]$ is equal to the Jacobson radical of $R[X]$. (Jacobson radical of a ring is the intersection of all the maximal ideals).

18. Let R be a commutative ring with $1 \neq 0$. Let $J \subset R$ be an ideal. Define $J[X] := \{a_0 + a_1X + \cdots + a_nX^n \mid a_i \in J \,\forall\, i\}$. Prove that $J[X]$ is an ideal of $R[X]$. Give an example of an ideal of $R[X]$ which is not of this form. Prove that the rings $R[X]/J[X]$ and $(R/J)[X]$ are isomorphic.

19. Let R be a commutative ring with $1 \neq 0$. An element $e \in R$ is called an *idempotent* if $e^2 = e$. Let R be a local ring. Prove that the only idempotents of R are 0 and 1. (A ring is called *local* if it has only one maximal ideal).

20. Let $R = \mathbb{Z}[i]$. Prove that the ideal $P =< 1 + i >$ is a prime ideal of R. Prove that $< 2 >= P^2$ in R.

21. Prove that the ring $\frac{\mathbb{Z}[i]}{<2+i>}$ is isomorphic to the ring $\mathbb{Z}/5\mathbb{Z}$.

22. Let R be a commutative ring with $1 \neq 0$. Let R be a ring and $a, b \in R$ be two comaximal elements, i.e., there exist $x, y \in R$ such that $xa + yb = 1$. Prove that for any $m, n \in \mathbb{N}$, one has a^m and b^n are also comaximal (i.e., there exist $\lambda, \mu \in R$ such that $\lambda a^m + \mu b^n = 1$).

23. Show that $X^2 + 1$ is a prime element in $\mathbb{Z}[X]$. Show that the elements $X^2 + 1$ and $X^6 + X^3 + X + 1$ are coprime in $\mathbb{Z}[X]$.

24. Let R be an integral domain. Let $a, b \in R$ be such that their gcd exists. If d, d' are both gcds of a, b, prove that d and d' are associates.

25. Let R be an integral domain such that $\gcd(a, b)$ exists for any $a, b \in R \setminus \{0\}$. Then prove that $\text{lcm}(a, b)$ exists for any $a, b \in R \setminus \{0\}$. (Hint: First prove that $\gcd(ma, mb) = m \gcd(a, b)$).

26. Let R be an integral domain such that $\text{lcm}(a, b)$ exists for any $a, b \in R \setminus \{0\}$. Then prove that $\gcd(a, b)$ exists for any $a, b \in R \setminus \{0\}$.

27. Let R be an integral domain such that $\gcd(a, b)$ exists for any $a, b \in R \setminus \{0\}$. If $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$, then prove that $\gcd(a, bc) = 1$.