**INDIAN STATISTICAL INSTITUTE**

Rings: Assignment II

MMath 1st year

Algebra I

**By a ring, we mean a commutative ring with 1. Also, we exclude the zero ring.**

1. Prove that $\mathbb{Z}[\sqrt{-2}]$, $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[\sqrt{3}]$ are all Euclidean domains.
2. Show that $\langle 2, 1+\sqrt{-17}\rangle$ is not a principal ideal in $\mathbb{Z}[\sqrt{-17}]$. Is this a prime ideal?
3. We just saw above that $\mathbb{Z}[\sqrt{-17}]$ is not a PID. Is it a UFD?
4. Let $R = \mathbb{Z}[\sqrt{-5}]$. Verify the following :
   (a) There are two distinct (i.e. non-associate) factorizations of 6 into irreducibles in $R$ : $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$.
   (b) The ideals $< 2, 1 \pm \sqrt{-5} >, < 3, 1 \pm \sqrt{-5} >$ are all prime ideals.
   (c) The following factorization of ideals into prime ideals holds:
   - $< 2 >=< 2, 1+\sqrt{-5} >^2$,
   - $< 3 >=< 3, 1+\sqrt{-5} >< 3, 1-\sqrt{-5} >$,
   - $< 1+\sqrt{-5} >=< 2, 1+\sqrt{-5} >< 3, 1+\sqrt{-5} >$,
   - $< 1-\sqrt{-5} >=< 2, 1+\sqrt{-5} >< 3, 1-\sqrt{-5} >$ .

   (d) Uniqueness of factorization is restored in (a) at the level of ideals.
   (e) The ideal $< 3, 1+\sqrt{-5} >$ is not a principal ideal.
5. Find gcd of $11+7i$ and $18-i$ in $\mathbb{Z}[i]$.
6. Show that $\frac{\mathbb{Z}[i]}{I}$ is a finte ring for any nonzero ideal $I$.
7. Let $R$ be an ID. Assume that:
   (a) any two $a, b \in R \setminus \{0\}$ have gcd, say, $d$ and there are $x, y \in R$ such that $d = ax + by$;
   (b) If $a_1, a_2, \cdots$ be nonzero elements of $R$ such that $a+i+1$ divides $a_i$ for $i = 1, 2, \cdots$, then there is $N > 0$ such that $a_n = u_n a_N$ for some unit $u_n \in R$ for all $n \geq N$.

   Show that $R$ is a PID.
8. Prove that $\frac{\mathbb{R}[X]}{\langle X^3+X\rangle}$ is a product of fields.
9. Let $m, n \in \mathbb{Z}$ be two nonzero elements and $d = \gcd(m, n)$ in $\mathbb{Z}$. Prove that $d$ is also a gcd of $m, n$ in $\mathbb{Z}[i]$.
10. Let $m, n \in \mathbb{Z}$ be two nonzero elements and let $m$ divide $n$ in $\mathbb{Z}[i]$. Does $m$ also divide $n$ in $\mathbb{Z}$?