INDIAN STATISTICAL INSTITUTE

Holiday Homework: Modules

MMath 1st year

Algebra I

By a ring, we mean a commutative ring with 1. Also, we exclude the zero ring.

1. From Dummit-Foote: Exercise 10.1: 1, 4, 5, 6, 7, 8, 9, 10, 11, 15.

2. From Dummit-Foote: Exercise 10.2: 3, 4, 5, 6, 7, 9, 10, 11, 12, 13.

3. An R-module is called *simple* if it is not the zero module and it has no proper submodule.

   (a) Prove that an R-module M is simple if and only if it is isomorphic (as an R-module) to $R/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of R. Show also that M is generated by any non-zero element.

   (b) Let M, N be simple R-modules. Prove that a module morphism $\phi : M \to N$ is either the zero morphism or an isomorphism. Conclude that if M is simple, $Hom_R(M, M)$ is a division ring.

4. Let M be an additive abelian group and $End(M)$ be the set of all group morphisms from M to M.

   (a) Show that $End(M)$ is a ring with identity (where addition is defined pointwise, and multiplication is composition of maps).

   (b) Show that M is an $End(M)$-module under the action:
   $$(f, m) \mapsto fm := f(m)$$

   (c) Let R be a ring and $\phi : R \to End(M)$ be a ring morphism. Show that M is an R-module under the action:
   $$(r, m) \mapsto rm := \phi(r)(m)$$

5. (Converse of 4(c)): Let R be a ring and M be an R-module. Show that there is a ring morphism $\phi : R \to End(M)$.

6. Let M be an R-module and A, B, C be its submodules such that
   $$A \subseteq B, \;\; A + C = B + C, \;\; A \cap C = B \cap C.$$
   Prove that $A = B$.

7. Let $f : M \to N$ be an R-module morphism. Let A be a submodule of M. Prove that $f^{-1}(f(A)) = A + Ker(f)$. Now let B be a submodule of N. Prove that $f(f^{-1}(B)) = B \cap Im(f)$. Also, show that
   $$f(A \cap f^{-1}(B)) = f(A) \cap B.$$

8. Let R be a commutative ring. Show that a map $f : R \times R \to R$ is an R-module morphism if and only if $f(x, y) = \alpha x + \beta y$ for some $\alpha, \beta \in R$.

9. Let $f : M \to N$ be an R-module morphism. Show that f can be expressed as the composition of three R-module morphisms (for appropriate R-modules): $f = g \circ h \circ k$, where g is surjective, h is an isomorphism, and k is injective.

10. Let R be an integral domain and $a \in R$ be non-zero. Let n be any positive integer. Prove that Ra^n/Ra^{n+1} is isomorphic to R/Ra (as R-modules).