INDIAN STATISTICAL INSTITUTE

Holiday Homework: Rings

MMath 1st year

Algebra I

By a ring, we mean a commutative ring with 1. Also, we exclude the zero ring.

1. From Dummit-Foote: Exercise 9.2: 1, 2, 3, 4, 5, 7, 8, 10.
2. From Dummit-Foote: Exercise 9.3: all of them.
3. From Dummit-Foote: All the examples after Eisenstein's criterion. Exercise 9.4: 1, 2, 3, 4, 9, 10, 11, 12, 13, 14, 17.
4. Let $f \in \mathbb{Z}[X]$ be a monic polynomial. Let $\alpha \in \mathbb{Q}$ be a root of f. show that $\alpha \in \mathbb{Z}$.
5. Let $f(X) \in F[X]$ be irreducible. Let Y be an indeterminate and $F(Y)$ denote the quotient field of $F[Y]$. Prove that $f(X)$ is also irreducible in $F(Y)[X]$.
6. Show that the polynomial $XY - ZW$ is irreducible in $\mathbb{C}[X, Y, Z, W]$.
7. Show that $f, g \in \mathbb{Z}[X]$ are comaximal in $\mathbb{Q}[X]$ if and only if the ideal $f\mathbb{Z}[X] + g\mathbb{Z}[X]$ contains an integer.

Supplementary problems from earlier topics:

8. Let $\varphi : \mathbb{R}[X, Y] \to \mathbb{R}[T]$ be the ring morphism defined by $\varphi(f(X, Y)) = f(T^2, T^3)$. Show that the kernel of φ is a principal ideal generated by $Y^2 - X^3$.
9. Identify the quotient ring $\mathbb{Z}[i]/\langle i - 2\rangle$.
10. Find the generator(s) of the kernels of the following ring morphisms:
    (a) $\mathbb{R}[X, Y] \to \mathbb{R}$ defined by $f(X, Y) \mapsto f(0, 0)$.
    (b) $\mathbb{R}[X] \to \mathbb{C}$ defined by $f(X) \mapsto f(2 + i)$.
    (c) $\mathbb{Z}[X] \to \mathbb{R}$ defined by $f(X) \mapsto f(1 + \sqrt{2})$.
    (d) $\mathbb{Z}[X] \to \mathbb{C}$ defined by $f(X) \mapsto f(\sqrt{3} + \sqrt{2})$.
    (e) $\mathbb{C}[X, Y, Z] \to \mathbb{C}[T]$ defined by $X \mapsto T$, $Y \mapsto T^2$, $Z \mapsto T^3$.
11. Let $\varphi : \mathbb{C}[X, Y] \to \mathbb{C}[T]$ be the ring morphism defined by $X \mapsto T + 1$, $Y \mapsto T^3 - 1$. Determine the kernel K of φ and show that any ideal I of $\mathbb{C}[X, Y]$ that contains K, is generated by two elements.
12. Let R be aring and $f(Y) \in R[Y]$. Prove that $\varphi : R[X, Y] \to R[X, Y]$ defined by $X \mapsto X + f(Y)$, $Y \mapsto Y$ is a ring automorphism of $R[X, Y]$.
13. Find all ring automorphisms of $\mathbb{Z}[X]$.
14. Determine the maximal ideals of: (a) $\dfrac{\mathbb{R}[X]}{\langle X^2 - 3X + 2\rangle}$; (b) $\dfrac{\mathbb{R}[X]}{\langle X^2 + X + 1\rangle}$