Practice problems for Class Test 2

① Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence such that there is a morphism $t : M'' \to M$ with $g \circ t = id_{M''}$. Prove that:

ⓐ t is injective.

ⓑ $M = t(M'') + \ker g$

ⓒ $t(M'') \cap \ker g = (0)$

Conclude that $M = t(M'') \oplus \ker g$

$$= t(M'') \oplus \operatorname{Im} f \simeq M'' \oplus M'.$$

(You may assume that $M = A \oplus B$ iff $M = A + B$ and $A \cap B = (0)$, where A, B are submodules of M)

~~(A) With the same assum~~

ⓓ Show directly ~~that~~ (without using a, b, c) that there is a morphism $s : M \to M'$ such that $s \circ f = id_{M'}$.

② Let $R = \mathbb{Z}/6\mathbb{Z}$, $P = \{\bar{0}, \bar{2}, \bar{4}\}$, $Q = \{\bar{0}, \bar{3}\}$ – ideals. Then check that $R = P \oplus Q$. This shows that P is a projective R-module. Show that P cannot be free.

③ Let n be a positive integer with $n = d_1 d_2$. Consider the sequence

$$0 \to d_2 \mathbb{Z}/n\mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\alpha} d_1 \mathbb{Z}/n\mathbb{Z} \to 0$$

where β is the inclusion map and α is multiplication by d_1.

Show that : ⓐ the sequence is exact.

ⓑ The sequence splits iff $\gcd(d_1, d_2) = 1$.

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(4) Let M be an R-module and $p : M \to M$ be a morphism such that $p^2 = p$. Prove that $M \simeq \ker p \oplus \operatorname{Im} p$.

(5) Let M and N be R-modules and $f : M \to N$, $g : N \to M$ be morphisms such that $f \circ g = id_N$. ~~and~~ Show that

$$M = \ker f \oplus \operatorname{Im} g .$$

(6) Let $M = M_1 \oplus M_2$, Show that $M_1 \simeq M/M_2$ and $M_2 \simeq M/M_1$.

(7) Let $M \xrightarrow{f} N \xrightarrow{g} P$ be a sequence of R-modules (not necessarily exact). Prove that

(a) $\operatorname{Im} f \cap \operatorname{Ker} g = f(\operatorname{Ker}(g \circ f))$

(b) $\operatorname{Im} f + \operatorname{Ker} g = g^{-1}(\operatorname{Im}(g \circ f))$.

(8) Let M, N be R-modules such that we have an exact sequence

$$0 \to M \xrightarrow{f} N \xrightarrow{g} R \to 0 .$$

Without using any result prove that this sequence splits.