① Let M be an R-module. Define

$$M[X] = \{ m_0 + m_1 X + \cdots + m_n X^n \mid m_i \in M,\ n \in \mathbb{N} \}.$$

Show that $M[X]$ is an ~~…~~ $R[X]$-module in a natural way.

Prove that $M[X] \simeq R[X] \otimes_R M$.

② Let P be a prime ideal in R. Show that $P[X]$ (as above) is a prime ideal in $R[X]$. If M is a maximal ideal of R, is $M[X]$ maximal in $R[X]$?

③ Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. If M' and M'' are finitely generated, show that so is M.

④ Let M be a finitely generated R-module and $\varphi : M \to R^n$ be a surjective morphism. Show that $\operatorname{Ker} \varphi$ is finitely generated.

⑤ Let $f : R \to S$ be a ring morphism and let N be an S-module. Regard N as an R-module ($r.n := f(r)n$). Now form the S-module $N_S = S \otimes_R N$. Show that the morphism $g : N \to N_S$ mapping y to $1 \otimes y$ is injective and that $g(N)$ is a direct summand of N_S.

⑥ Let M be a Noetherian R-module. Show that $M[X]$ (see Ex 1) is a Noetherian $R[X]$-module.

⑦ Let R be a ring. Define $R_n[X] := \{$all polynomials of degree $\leq n\}$. Show that $R_n[X]$ is an R-module $\forall\, n \geq 1$, and also,

$$R_n[X] \simeq R_{n-1}[X] \text{ as } R\text{-modules.}$$

More problems 5/11/24

1. Let M be an R-module. Define
$$\operatorname{ann}(M) := \{a \in R \mid am = 0 \ \forall m \in M\}.$$
Show that: $\operatorname{ann}(M)$ is an ideal of R.
Further, M is an R/I-module for any ideal $I \subseteq \operatorname{ann}(M)$.

2. Let M be a Noetherian R-module. Prove that $R/\operatorname{ann}(M)$ is a Noetherian ring.

3. Show that the R-submodules of M are in one to one correspondence with the $R/\operatorname{ann}(M)$-submodules of M.

4. Give an example of a ring R and a finitely gen'd R-module M such that M has a submodule N that is not finitely generated.

5. Give example of R-modules M_1, M_2 and submodules N_1 of M_1, N_2 of M_2 such that $N_1 \otimes_R N_2$ is <u>not</u> a submodule of $M_1 \otimes_R M_2$.

6. Let R, S be commutative rings, P be an R-S bimodule, M be an R-module and N be an S-module.
Prove that $(M \otimes_R P) \otimes_S N \simeq M \otimes_R (P \otimes_S N)$

7. Let R be a commutative ring, $I \subseteq R$ be an ideal, M, N be R-modules. Prove that
$$M/IM \otimes_{R/I} N/IN \simeq (M \otimes_R N) / I(M \otimes_R N).$$