Assignment 1

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a linear map.
   (a) How does f look like?
   (b) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous such that f satisfies $f(x+y) = f(x) + f(y)$. How does f look like?

2. For each $p \in [0, \infty]$, we have the norms $\|\cdot\|_p$ on $\mathbb{R}^n$. Let

$$B_p = \{x \in \mathbb{R}^2 \colon \|x\|_p \leq 1\}$$

   Draw $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_\infty$ and observe how $\mathcal{B}_p$ behaves as $p \to \infty$.

3. Prove that any finite dimensional nls is complete.

4. (a) Prove that any linear map from $(\mathbb{R}^n, \|\cdot\|)$ to $(\mathbb{R}^m, \|\cdot\|')$ is continuous. (Hint: We have already seen that if $\|.\|_2$ is the Euclidean norm, then

$$\|T(x)\|_2 \leq \|T\|_E \|x\|_2.$$

   Use this to show that T is continuous.)
   (b) Prove that any linear map between any two finite dimensional normed linear spaces is continuous.
   (c) Prove that if $(V, \|\cdot\|)$ and $(W, \|\cdot\|')$ are finite dimensional normed linear spaces, then $\mathcal{L}(V, W)$ is a normed linear space under the operator norm.

5. Suppose $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are finite dimensional nls and T_1, T_2 are linear maps from V to W such that

$$T_1(x) = T_2(x) \ \forall x \in \{y \in \mathbb{R}^n \colon \|y\|_V \ < \delta\},$$

   where δ is a positive number. Prove that $T_1 = T_2$.

6. Prove that any subspace of a finite dimensional nls is closed.

7. Suppose W is a subspace of a finite dimensional nls which is open. What can you say about W?

8. Suppose $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear isomorphism. Then what can you say about m and n?

9. Suppose V is a finite dimensional vector space of dimension n such that $T^2 = 0$. Prove that

$$\operatorname{rank}(T) \leq \frac{n}{2}.$$

10. Let $< \cdot, \cdot >: V \times V \to \mathbb{R}$ be an inner product on a vector space V. Prove that $< \cdot, \cdot >$ is continuous in both the variables, i.e., if $x_0, y_0 \in V$, then the maps

$$I_{x_0,1} \colon V \to V,\ I_{x_0,1}(y) = < x_0, y >$$

and

$$I_{y_0,2} \colon V \to V,\ I_{y_0,2}(y) = < y, y_0 >$$

are continuous.

11. Prove that

$$\|f\|_{\sup} = \sup_{x \in [0,1]} |f(x)|$$

defines a norm on $C[0, 1]$.

12. Prove that

$$\|f\|_{\sup} = \sup_{x \in \mathbb{R}} |f(x)|$$

defines a norm on $C_0(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \text{ continuous such that } \lim_{|x| \to \infty} |f(x)| = 0\}$.

13. Let $\mathcal{R}[a, b]$ denotes the set of all Riemann integrable functions on $[a, b]$

    (a) Does $\|f\| = \int_a^b |f(x)|dx$ define a norm on $\mathcal{R}[a, b]$?

    (b) Does $< f, g >= \int_a^b f(x)g(x)dx$ define an inner product on $C[a, b]$?

14. Let $(V, \| \cdot \|_V)$, $(W, \| \cdot \|_W)$ and $(X, \| \cdot \|_X)$ be finite dimensional nls.

    (a) If $T \in \mathcal{L}(V, W)$, prove that

$$\begin{aligned} \|T\|_{op} &= \sup\{\|T(x)\|_W \colon \|x\|_V = 1\} \\ &= \sup\{\|T(x)\|_W \colon \|x\|_V < 1\} \\ &= \inf\{K : K \geq 0 \text{ and } \|T(x)\|_W \leq K\|x\|_V\} \end{aligned}$$

    (b) Prove that $\|T(x)\|_W \leq \|T\|_{op}\|x\|_V \ \forall x \in V$.

    (c) If S is a bounded subset of V in $\| \cdot \|_V$, and $T \in \mathcal{L}(V, W)$,, prove that $T(S)$ is bounded subset of W in $\| \cdot \|_W$.

    (d) If $T_1 \in \mathcal{L}(V, W)$ and $T_2 \in \mathcal{L}(W, X)$, prove that $\|T_2T_1\|_{op} \leq \|T_1\|_{op}\|T_2\|_{op}$.

    (e) Prove that if $V = W$ and $\| \cdot \|_V = \| \cdot \|_W$, then $\|I\|_{op} = 1$.

    (f) Now suppose $< \cdot, \cdot >$ be inner products on V which induces $\| \cdot \|_V$.
    An element $T \in \mathcal{L}(V, V)$ is called an orthogonal projection if $T^2 = T$ and $T^* = T$; i.e. $< Tv, w >=< v, Tw >$, $\forall v, w \in V$.
    Prove that the operator norm of a non-zero orthogonal projection is 1.

15. Prove that a subset K of a finite dimensional nls is compact if and only if it is closed and bounded.