Assignment 2

1. Examine the continuity of the following functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ at the point $(0,0)$:

   (a)
   $$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$$

   (b)
   $$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$$

   (c)
   $$f(x,y) = \begin{cases} \frac{|x|}{y^2} e^{\frac{-|x|}{y^2}} & \text{if } y \neq 0 \\ 0 & \text{o.w.} \end{cases}$$

   (d)
   $$f(x,y) = \begin{cases} x \sin\frac{1}{y} + y \sin\frac{1}{x} & \text{if } xy \neq 0 \\ 0 & \text{o.w} \end{cases}$$

2. Consider the following funcion $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by:
   $$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$$
   Let $m \in \mathbb{R}$. Prove that $\lim_{x\to 0} f(x, mx) = f(0,0) = 0$ but f is not continuous at $(0,0)$.

3. If $x = (x_1, \cdots x_n)$ denotes an element of $\mathbb{R}^n$, prove that
   $$\|x\| \leq \sum_{i=1}^{n} |x_i| \,.$$

4. Suppose x and y belong to $\mathbb{R}^n$. When does equality hold in the triangle inequality
   $$\|x+y\| \leq \|x\| + \|y\|?$$

5. Suppose $\mathbb{R}^n$ is equipped with the usual inner product and the usual norm. A linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is called norm preserving if $\|Tx\| = \|x\|$. T is called angle-preserving if $\langle Tx, Ty\rangle = \langle x, y\rangle$.

(a) Prove that T is norm-preserving if and only if T is angle-preserving.

(b) Prove that a norm-preserving linear map T as above is one-one and moreover, T^{-1} is also norm-preserving.

6. Suppose $\mathbb{R}^n$ is equipped with the usual inner product and the usual norm. We will let $(\mathbb{R}^n)^*$ denote the vector space $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$. More generally, if V is a real vector-space, V^* will denote the vector space $\mathcal{L}(V, \mathbb{R})$.

   Now for $x \in \mathbb{R}^n$, we define an element ϕ_x in $(\mathbb{R}^n)^*$ by the formula

   $$\phi_x(y) = \langle x, y \rangle \,.$$

   (a) Check that ϕ_x is indeed a linear map from $\mathbb{R}^n$ to $\mathbb{R}$.

   (b) Prove that

   $$\psi : \mathbb{R}^n \to (\mathbb{R}^n)^*$$

   is a vector space isomorphism. This shows that given any functional ϕ on $\mathbb{R}^n$, there exists a unique y in $\mathbb{R}^n$ such that for all x in $\mathbb{R}^n$,

   $$\phi(x) = \langle x, y \rangle \,.$$

7. Suppose $\mathbb{R}^n$ is equipped with the usual inner product and the usual norm. Two vectors x, y are called orthogonal if $\langle x, y \rangle = 0$. If x and y are orthogonal, prove that

   $$\|x + y\|^2 = \|x\|^2 + \|y\|^2 \,.$$

8. (a) If (X, d) is a metric space and $Y \subseteq X$, prove that (Y, d) is a metric space.

   (b) If X and Y are as in part a, prove that a set U is open in (Y, d) if and only if $U = Y \bigcap V$, where V is an open set in X.

   (c) Let $X \subseteq \mathbb{R}^n$. A function $f\colon X \to \mathbb{R}^m$ is said to be continuous at $a \in X$ if given $\epsilon > 0$, there exists $\delta > 0$ such that for all $yinX$ such that $||y - a|| < \delta$, we have $||f(y) - f(a)|| < \epsilon$.
   f is said to be continuous on X if f is continuous at a, for all $a \in X$.
   Prove that f is continuous if and only if $\forall\ W \subseteq \mathbb{R}^m$ open, we have $f^{-1}(W) = X \bigcap V$ for some V open in $\mathbb{R}^n$.

9. Give an example of an open set U in $\mathbb{R}^2$ such that U is not of the form $V \times W$, where V and W are subsets of $\mathbb{R}$.

10. Examine whether the following subsets of $M_n(\mathbb{R})$ are open, closed, compact, connected subsets of $M_n(\mathbb{R})$:

    (a) $Gl_n(\mathbb{R}) = \{A \in M_n(\mathbb{R})\colon A \text{ is invertible}\}$

    (b) $O_n(\mathbb{R}) = \{A \in M_n(\mathbb{R})\colon A^t A = A A^t = I_n\}$

    (c) $S^1 = \{(x, y) \in \mathbb{R}^2\colon x^2 + y^2 = 1\}$