Assignment 3

1. If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ is differentiable at a, then prove that there exists a **unique** linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0}\frac{||f(a+h) - f(a) - T(a)(h)||}{||h||} = 0.$$

2. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and $v \in \mathbb{R}^m$. Define $f : \mathbb{R}^n \to \mathbb{R}^m$ by $f(x) = Ax+v$. Compute the derivative of f.

3. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a function satisfying $|f(x)| \leq ||x||^2$. Prove that f is differentiable at 0.

4. Compute the derivative of the following functions:

   (a) $f(x, y, z) = x^y$,

   (b) $f(x, y, z) = \sin(x \sin(y \sin z))$.

5. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Compute the derivative of the following functions:

   (a) $f(x, y) = \int_a^{x+y} g(t)dt$,

   (b) $f(x, y) = \int_a^{x.y} g(t)dt$.

6. Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is a bilinear function. Prove the following statements:

   (a) $\lim\limits_{(h,k)\to(0,0)} \frac{||f(h,k)||}{||(h,k)||} = 0$.

   (b) $Df(a, b)(x, y) = f(a, y) + f(x, b)$.

   (c) Cross check that if $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined as $p(x, y) = x.y$, then (b) recovers the formula for Dp derived during one of the lectures.

7. Let $< \ , \ >$ denote the standard Euclidean inner product on $\mathbb{R}^n$. We define

$$IP : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \text{ by } IP(x, y) =< x, y > .$$

   (a) Compute $D(IP)(a, b)$.

   (b) If $f, g : \mathbb{R} \to \mathbb{R}^n$ are differentiable functions and $h : \mathbb{R} \to \mathbb{R}$ is defined by

$$h(t) =< f(t), g(t) >,$$

   then show that

$$h'(a) =< Df(a)(1), g(a) > + < f(a), Dg(a)(1) > .$$

(c) If $f : \mathbb{R} \to \mathbb{R}^n$ is a differentiable function such that $||f(t)|| = 1$ for all t, then show that

$$< Df(t)(1), f(t) >= 0.$$

8. Let $f : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \to \mathbb{R}^p$ be a multilinear map.

(a) If $i \neq j$ and $h = (h_1, \cdots h_k) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$, then prove that

$$\lim_{h \to 0} \frac{||f(a_1, a_2, \cdots h_i, \cdots h_j, a_{j+1}, \cdots a_k)||}{||h||} = 0.$$

(**Hint:** If $g(x, y) = f(a_1, \cdots x, \cdots y, \cdots a_k)$, then g is bilinear.)

(b) Prove that

$$Df(a_1, \cdots a_k)(x_1, \cdots x_k) = \sum_{i=1}^{k} f(a_1, \cdots a_{i-1}, x_i, a_{i+1}, \cdots a_k).$$

9. Recall that $\det : \mathbb{R}^n \times \cdots \mathbb{R}^n \to \mathbb{R}$ is a multilinear map.

(a) Prove that det is differentiable and

$$D(\det)(a_1, \cdots a_n)(x_1, \cdots x_n) = \sum_{i=1}^{n} \det \begin{pmatrix} a_1 \\ \cdots \\ x_i \\ \cdots \\ a_n \end{pmatrix}.$$

(b) If $a_{ij} : \mathbb{R} \to \mathbb{R}$ are differentiable functions and $f(t) = \det(a_{ij}(t))$, show that

$$f'(t) = \sum_{j=1}^{n} \det \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \cdots & \cdots & \cdots \\ a'_{j1}(t) & \cdots & a'_{jn}(t) \\ \cdots & \cdots & \cdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}.$$

10. (a) For an $n \times n$ matrix $A = (a_{ij})$, define $\mathrm{Tr}(A) = \sum_i a_{ii}$. Prove that Tr is a linear map from $M_n(\mathbb{R})$ to $\mathbb{R}$ satisfing the equation $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

(b) Now suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map and let $\mathcal{B}$ be a fixed basis of the vector space $\mathbb{R}^n$. Define $\mathrm{Tr}(T)$ to be the trace of the matrix of T in the basis $\mathcal{B}$. Prove that $\mathrm{Tr}(T)$ is well-defined, i.e, the definition of $\mathrm{Tr}(T)$ is independent of the choice of the basis $\mathcal{B}$.

(c) If $\det : M_n(\mathbb{R}) \to \mathbb{R}$ denotes the determinant function, prove that

$$D(det)(I)(B) = \mathrm{Tr}(B).$$

11. If $f : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is defined by $f(A) = A^{-1}$, prove that for all B in $M_n(\mathbb{R})$,

$$Df(A)(B) = -A^{-1}BA^{-1}.$$

12. Consider the function $f : \mathbb{R}^n \setminus \{0\},\ f(x) = ||x||$. Prove that for all x, y in $\mathbb{R}^n$, we have

$$Df(x)(y) = \frac{< x, y >}{||x||}.$$

13. Consider the following function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$$

Prove that all directional derivatives of f at $(0,0)$ exist but f is not even continuous at $(0,0)$.

14. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, does there exist $c \in \mathbb{R}^n$ such that $Df(c)(u) > 0$ for all $u \in \mathbb{R}^n$?

15. Suppose $x_0 \in \mathbb{R}^n$ and $f : B(x_0, r) \to \mathbb{R}$ be a differentiable function such that there exists $u \in \mathbb{R}^n, u \neq 0$ such that $Df(x)(u) = 0$ for all $x \in B(x_0, r)$. What can you say about f?

16. Suppose U is an open set in $\mathbb{R}^n$. A function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is said to have C^1-directional derivatives if all directional derivatives exist and are continuous, i.e, for all $v \in \mathbb{R}^n$, the map $U \to \mathbb{R}^m,\ x \mapsto Df(x)(v)$ is continuous.

    Prove that f is C^1 if and only if f has C^1-directional derivatives.

17. For an open set U in $\mathbb{R}^n$, let $f : U \to \mathbb{R}$ be a differentiable function such that $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_j \partial x_i}$ are continuous on an open set containing a, then prove that

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a).$$

    (Hint: Let $a = (a_1, \cdots a_n)$.

    We convert the problem into a 2-variable one.

    Find an open set U in $\mathbb{R}^2$ such that the following function is defined:

$$g : U \to \mathbb{R},\ g(x,y) = f(a_1, a_2, \cdots a_{i-1}, x, a_{i+1}, \cdots a_{j-1}, y, a_{j+1}, \cdots a_n).$$

    Observe that it is enough to prove that

$$\frac{\partial^2 g}{\partial x \partial y}(a_i, a_j) = \frac{\partial^2 g}{\partial y \partial x}(a_i, a_j).$$

Now define another two variable function F which can be written in two ways:

$$F(h,k) = [g(a_i + h, a_j + k) - g(a_i + h, a_j)] - [g(a_i, a_j + k) - g(a_i, a_j)],$$

$$F(h,k) = [g(a_i + h, a_j + k) - g(a_i, a_j + k)] - [g(a_i + h, a_j) - g(a_i, a_j)].$$

Now apply the one-variable mean-value theorem four times, i.e, twice for each equation.)

18. Find the partial derivatives of the following functions:

    (a) $f(x, y, z) = x^y$,

    (b) $f(x, y, z) = \sin(x \sin(y \sin z))$.