Assignment 4

1. Suppose U is an open set in $\mathbb{R}^n$ and $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is a differentiable function such that f attains a **local maxima or minima** at the point x. Using the function $g(t) = f(x+tv)$ defined on a suitable open interval containing 0, prove that $\nabla f(x) = 0$, i.e, $\frac{\partial f}{\partial x_i}(x) = 0$ for all $i = 1, 2, \cdots n$.

2. Suppose U is an open set in $\mathbb{R}^n$ and $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is a differentiable function. Let x_0 be a fixed element in U where at least one partial derivative of f is not equal to zero. Let us denote by $S^{n-1}$the set

$$S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

Define a function

$$g : S^{n-1} :\to \mathbb{R},\ g(v) = |D_v f(x_0)|\,.$$

Prove that g attains its maxima at the points $\pm \frac{\nabla f(x_0)}{\|\nabla f(x_0)\|}$.

In other words, the direction v in which $|D_v f(x_0)|$ is maximum is along $\nabla f(x_0)$.

3. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined as $f(t) = (\cos(t), \sin(t))$. Prove that there exist $x, y \in [0, 2\pi]$ such that the equation

$$f(y) - f(x) = Df(z)(y - x)$$

cannot hold for any $z \in [x, y]$.

4. Suppose U is an open set in $\mathbb{R}^n$ which is convex, i.e, for any a, b in U, the set $\{tx + (1-t)y : 0 \le t \le 1\}$ is contained in U.

   (a) Prove that if $f : U \to \mathbb{R}^m$ is a C^1-function, then

   $$\sup_{0 \le t \le 1} ||Df(x + t(y-x))||_{\text{op}} < \infty.$$

   (b) Suppose f is a real-valued differentiable function defined on an open set U in $\mathbb{R}^n$. If x, y belonging to U is such that $L(x, y) \subseteq U$, then prove that

   $$f(y) - f(x) = \langle \nabla f(z), y - x \rangle$$

   for some $z \in L(x, y)$.

   (c) Suppose $U \subseteq \mathbb{R}^n$ is a convex open set and $f : U \to \mathbb{R}^m$ is a differentiable function such that all partial derivatives of f are bounded on U. Prove that f is Lipschitz on U, i.e,
   there exists a **real number** $A > 0$ such that for all $x, y \in U$,

   $$\|f(y) - f(x)\| \le A \|y - x\|\,.$$

(d) Prove that if $f : U \to \mathbb{R}^m$ is a differentiable function and T is any linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, then for $x, y \in U$ such that $L(x, y) \subseteq U$, we have

$$||f(y) - f(x) - T(y - x)|| \leq ||y - x|| \text{sup}_{0 \leq t \leq 1} ||Df(x + t(y - x)) - T||_{\text{op}}.$$

5. If V and W are finite dimensional vector spaces, prove that the dimension of the vector space of all k-multilinear maps from V to W is equal to $\dim(V)^k \dim(W)$.

6. Suppose U is an open convex set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ be a C^2-function. Prove that for all a in U and h in $\mathbb{R}^n$ such that $\|h\|$ is sufficiently small,

$$f(a + h) = f(a) + \langle \nabla f(a), h \rangle + \frac{1}{2} D^2 f(a)h + \|h\|^2 E(h),$$

where E is a real-valued function defined on an open set containing zero such that $\|E(h)\| \to 0$ as $\|h\| \to 0$.

7. Prove that the map

$$f : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}), \ f(A) = A^{-1}$$

is C^∞.