Assignment 5

1. Suppose $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, where U is an open set in $\mathbb{R}^n$. Let $p \in U$ and $v \in \mathbb{R}^m$. Prove that
$$Df(x)(v) = \frac{d}{dt}|_{t=0} f(\gamma(t)),$$
where γ is a smooth curve passing through p with velocity v.

2. Suppose $X \in M_n(\mathbb{R})$. Prove that
$$\gamma : \mathbb{R} \to M_n(\mathbb{R}), \gamma(t) = e^{tX}$$
is a curve passing through I with velocity X.

3. Prove that for any $X \in M_n(\mathbb{R})$,
$$\frac{d}{dt}|_{t=0} \det(e^{tX}) = \mathrm{Tr}(X).$$
(**Hint:** Use the fact that for any $X \in M_n(\mathbb{R})$, $D(\det)(I)(X) = \mathrm{Tr}(X)$.)

4. Prove that for any $X \in M_n(\mathbb{R})$,
$$\det(e^X) = e^{\mathrm{Tr}(X)}.$$
(**Hint:** Consider the function $g(t) = \det(e^{tX})$. Note that $g(0) = I$. Since $e^{(s+t)X} = e^{sX}e^{tX}$, observe that we can write
$$g'(s) = g(s)\frac{d}{dt}|_{t=0} \det(e^{tX}).$$
Now solve this differential equation with the initial condition $g(0) = I$.)

5. We had found out a very nice formula for $D(\det)(I)(X)$, namely, $D(\det)(I)(X) = \mathrm{Tr}(X)$. Now if A is an arbitrary element of $\mathrm{GL}_n(\mathbb{R})$, does there exist a nice formula for $D(\det)(A)(X)$? This exercise answers this question.

   Prove that for all $A \in \mathrm{GL}_n(\mathbb{R})$ and for all $X \in M_n(\mathbb{R})$,
$$D(\det)(A)(X) = \det(A)\mathrm{Tr}(A^{-1}X).$$
(**Hint:** Observe that $\gamma(t) = Ae^{tA^{-1}X}$ is a curve passing through A with velocity X. Next, you will need to use the equation $\det(e^X) = e^{\mathrm{Tr}(X)}$.)