Assignment 6

1. If U and V are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and $\phi : U \to V$ is a diffeomorphism, prove that $m = n$. This says that if $m \neq n$, then an open subset of $\mathbb{R}^n$ cannot be diffeomorphic to an open subset of $\mathbb{R}^n$.

2. Consider the function
$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x, y) = (e^x \cos(y),\ e^x \sin(y)).$$
Show that the Jacobian of f is not zero at any point of $\mathbb{R}^2$. Thus, by IMT, any point of $\mathbb{R}^2$ has a neighborhood in which f is one-one. Nevertheless, prove that f is not one-one on $\mathbb{R}^2$.

3. Define $f\colon \mathbb{R}^5 \to \mathbb{R}^2$, given by $f = (f_1, f_2)$ where $f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1} + x_2y_1 - 4y_2 + 3$ and $f_2(x_1, x_2, y_1, y_2, y_3) = x_2 \cos(x_1) - 6x_1 + 2y_1 - y_3$.
   (a) Show that $f(0, 1, 3, 2, 7) = (0, 0)$
   (b) Show that $\exists$ a C^1 map g defined on a neighbourhood of $(3, 2, 7)$ such that $g(3, 2, 7) = (0, 1)$ and $f(g(y), y) = (0, 0)$.
   (c) compute $Dg(3, 2, 7)$.

4. Using the implicit function theorem (and not otherwise), show that the system of equations:
$$3x + y - z + u^2 = 0$$
$$x - y + 2z + u = 0$$
$$2x + 2y - 4z + 2u = 0$$
has a local solution for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x.

5. Define $f\colon \mathbb{R}^3 \to \mathbb{R}$ by
$$f(x, y_1, y_2) = x^2y_1 + e^x + y_2$$
Show that $\frac{\partial f}{\partial x}(0, 1, -1) \neq 0$ and there exists a differentiable function g in a neighborhood of $(1, -1)$ in $\mathbb{R}^2$ so that $g(1, -1) = 0$ and $f(g(y_1, y_2), y_1, y_2) = 0$. Moreover find $\frac{\partial g}{\partial y_1}(1, -1)$ and $\frac{\partial g}{\partial y_2}(1, -1)$.

6. If S is a regular k-level surface in $\mathbb{R}^{n+k}$, k is called the dimension of S and n is called the codimension of S. For each of the following examples, determine whether the set $f^{-1}(0)$ is a regular surface. If your answer is yes, then also determine the dimension and the codimension.

(a) $f(x,y,z) = x^2 + y^2 + z^2 - 1$
(b) $f(x,y,z) = x^2 - y^2 - z^2$

7. Prove that the following are examples of regular surfaces. Also compute their dimension and codimension.

   (a) (the 2-torus)
   $$\mathbb{T}^2 = f^{-1}(1,1),$$
   where $f : \mathbb{R}^4 \to \mathbb{R}^2$ is defined by
   $$f(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2, x_3^2 + x_4^2).$$

   (b) (the n-torus)
   $$\mathbb{T}^n = f^{-1}(1, \cdots 1)$$
   where $f : \mathbb{R}^{2n} \to \mathbb{R}^n$ is defined by
   $$f(x_1, x_2, \cdots x_{2n-1}, x_{2n}) = (x_1^2 + x_2^2, \cdots x_{2n-1}^2 + x_{2n}^2).$$
   Also prove that $\mathbb{T}^n$ is the n-fold Cartesian product of S^1.

   (c) (the $(n-1)$ sphere in $\mathbb{R}^n$)
   $$S^{n-1} = f^{-1}(1)$$
   where $f : \mathbb{R}^n \to \mathbb{R}$ is defined by
   $$f(x_1, \cdots x_n) = \sum_{i=1}^{n} x_i^2.$$

8. Prove that $\mathbb{R}^n \times \{0\}$ is an n-manifold in $\mathbb{R}^{n+1}$.

9. Prove that $\mathrm{GL}_n(\mathbb{R})$ is a manifold in $\mathbb{R}^{n^2}$. What is its dimension?

10. (a) Recall that the derivative of the function $\det : GL_n(\mathbb{R}) \to \mathbb{R}$ is given by $D(\det)(A)(X) = \det(A)\mathrm{Tr}(A^{-1}X)$.
      i. Compute the dimension of the vector space $\mathrm{Ker}(D(\det))(I)$, where I denotes the identity matrix in $M_n(\mathbb{R})$.
      ii. Show that $\mathrm{SL}_n(\mathbb{R}) := \{A \in M_n(\mathbb{R}) : \det(A) = 1\}$ is a regular $n^2 - 1$-level surface in $\mathbb{R}^{n^2}$.

    (b) Prove that $O(n)$ (i.e, the set of all $n \times n$ real orthogonal matrices) is a manifold of dimension $\frac{n(n-1)}{2}$ in $M_n(\mathbb{R})$.
    (**Hint:** Let S_n denote the vector space of all $n \times n$ real symmetric matrices. Consider the function $f : M_n(\mathbb{R}) \to S_n$ defined by $f(A) = AA^t$.)

11. Suppose k, l are positive integers such that M is a k-manifold in $\mathbb{R}^n$ and moreover, M is an l-manifold in $\mathbb{R}^n$. Prove that $k = l$.

12. Suppose M and N are k-manifolds in $\mathbb{R}^n$ and $f : M \to N$ is a smooth function such that for all p in M, the linear map

$$Df(p) : T_pM \to T_{f(p)}N$$

is a vector space isomorphism. Then prove that if $p \in M$, there exists an open set V of M containing p which is diffeomorphic to an open set of N containing $f(p)$.

13. Let us recall the statement of the implicit function theorem:

    Suppose $U \subseteq \mathbb{R}^n$ is an open set and $f : U \to \mathbb{R}^m$ is a smooth function. Moreover, assume that there exist $(x_0, y_0) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$ such that $f(x_0, y_0) = 0$ and $D_{\mathbb{R}^m} f(x_0, y_0)$ is invertible.

    Then I.F.T. states that there exists an open set V in $\mathbb{R}^{n-m}$ containing x_0, an open set W in $\mathbb{R}^m$ containing y_0 and a smooth map $g : V \to W$ such that $D_{\mathbb{R}^m} f(x, y)$ is invertible for all $(x, y) \in V \times W$ and

$$\{(x, y) \in V \times W : f(x, y) = 0\} = \{(x, g(x)) : x \in V\}.$$

    We also computed an expression for $Dg(x_0)$.

    (a) In the notations as above, prove that the set

$$M = \{(x, y) \in U : f(x, y) = 0\}$$

    is an $n - m$-manifold in $\mathbb{R}^n$.

    (b) Prove that

$$T_{(x_0,y_0)}M = \{(v, Dg(x_0)(v)) : v \in \mathbb{R}^{n-m}\}.$$

    Thus, even without knowing the function g explicitly, the implicit function theorem helps us to understand the tangent space to the manifold M. This follows from the fact that we have a formula for $Dg(x_0, y_0)$ in terms of the function f from the implicit function theorem.

14. (*) We have seen that if p belongs to an open set U in $\mathbb{R}^n$, then T_pU can be identified with the set of all derivations of $C^\infty(p)$. We can go one step further, in the context of vector fields.

    Suppose U is an open set in $\mathbb{R}^n$. An $\mathbb{R}$-linear map $\delta : C^\infty(U) \to C^\infty(U)$ is called a derivation of $C^\infty(U)$ if for all $f, g \in C^\infty(U)$,

$$\delta(f.g) = \delta(f).g + f.\delta(g).$$

    The set of all derivations of $C^\infty(U)$ is denoted by the symbol $\mathrm{Der}(C^\infty(U))$. The goal of this exercise is to show that $\mathcal{X}(U) = \mathrm{Der}(C^\infty(U))$.

(a) We will need the following result, called the existence of partition of unity (See Theorem 3.11 of Spivak for a proof). We recall that the support of a real valued function defined on a topological space is

$$\operatorname{supp}(f) := \overline{\{x \in \operatorname{Dom}(f) : f(x) \neq 0\}}.$$

Theorem

Let $A \subseteq \mathbb{R}^n$ and let $\{U_i\}_{i\in I}$ be an open cover of A. Then there exists a collection $\{\phi_i : i \in I\}$ of smooth functions on an open subset of $\mathbb{R}^n$ containing A satisfying the following conditions:

i. For all $x \in A$, $0 \leq \phi_i(x) \leq 1$.

ii. For all $x \in A$, there exists V open in $\mathbb{R}^n$ containing x such that all but finitely many ϕ_i are zero on V.

iii. For all $x \in A$,

$$\sum_i \phi_i(x) = 1.$$

Note that this equation makes sense by the previous point.

iv. For all $i \in I$, $\operatorname{supp}(\phi_i) \subseteq U_i$.

The collection $\{\phi_i : i \in I\}$ is called a partition of unity subordinate to the cover $\{U_i : i \in I\}$.

As an application of the theorem on partition of unity, prove the following statement:

Suppose $A \subseteq \mathbb{R}^n$ is closed and U be an open set in $\mathbb{R}^n$ such that $A \subseteq U$. Then prove that there exists a real-valued smooth function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that $\psi(x) = 1$ for all $x \in A$, $\operatorname{supp}(\psi) \subseteq U$ and $0 \leq \psi(x) \leq 1$.

(b) Given an element f of $C^\infty(U)$ and X in $\mathcal{X}(U)$, we can define a real-valued function Xf on U by the formula

$$(Xf)(p) = X_p(f).$$

Here, the element X_p of $T_p(U)$ is viewed as an element of $\operatorname{Der}(C^\infty(p))$ and f is viewed as an element of $C^\infty(p)$ so that $X_p(f)$ makes sense.

Prove that Xf is a smooth function on U.

(c) Suppose $X \in \mathcal{X}(U)$. Then prove that the map

$$C^\infty(U) \to C^\infty(U), f \mapsto Xf$$

is a derivation of $C^\infty(U)$. Thus, $\mathcal{X}(U)$ is a subset of $\operatorname{Der}(C^\infty(U))$.

(d) Finally , prove that $\mathcal{X}(U) = \operatorname{Der}(C^\infty(U))$. This can be done in three steps:

i. Prove that if $X, Y \in \mathcal{X}(U)$ are such that $X(f) = Y(f)$ for all $f \in C^\infty(U)$, then $X = Y$.

ii. Suppose $\delta \in \mathrm{Der}(C^\infty(U))$ and $f \in C^\infty(U)$ is such that $f(x) = 0$ for all x on an open subset V of U. Prove that $\delta(f)(y) = 0$ for all $y \in V$.
(**Hint:** By the problem in a), observe that there exists an open set W in V such that $p \in W$ and a smooth function g on U such that $g = 1$ on W and $g = 0$ outside V.)

iii. Prove that $\mathcal{X}(U) = \mathrm{Der}(C^\infty(U))$. i.e, if $\delta \in \mathrm{Der}(C^\infty(U))$, then there is an unique element X in $\mathcal{X}(U)$ such that for all $f \in C^\infty(U)$, $\delta(f) = X(f)$.

(e) Let $X, Y \in \mathcal{X}(U)$. Define a map $[X,Y] : C^\infty(U) \to C^\infty(U)$ by the formula

$$[X,Y](f) = X(Yf) - Y(Xf).$$

This means that for all $p \in U$,

$$[X,Y](f)(p) = X_p(Yf) - Y_p(Xf).$$

Prove that $[X,Y]$ is a vector field on U. Moreover, write $[X,Y]$ as a $C^\infty(U)$-linear combination of the vector fields $\frac{\partial}{\partial x_i}$.

15. Suppose M is a k-manifold in $\mathbb{R}^n$. A vector field X on M is called non-vanishing on M if $X_p \neq 0$ for all $p \in M$. A vector field X on a k-manifold M in $\mathbb{R}^n$ is called a unit vector field if $< X(p), X(p) >= 1$ for all $p \in M$.

(a) Prove that there exists a non-vanishing tangent vector field on M if and only if there exists a unit tangent vector field on M.

(b) Prove that there exists a non-vanishing normal vector field on M if and only if there exists a unit normal vector field on M.

(c) Prove that on a connected regular level n-surface in $\mathbb{R}^{n+1}$, there exist exactly two unit normal vector fields.

(d) On a connected regular level k-surface in $\mathbb{R}^{n+k}$, how many unit normal vector fields can you think of?

(e) Suppose $k \geq 1$ and $n = 2k-1$. Consider the n-manifold $M = S^n$ inside $\mathbb{R}^{n+1}$. Prove that

$$X = (-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}) + (-x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}) + \cdots + (-x_{2k}\frac{\partial}{\partial x_{2k-1}} + x_{2k-1}\frac{\partial}{\partial x_{2k}})$$

defines a nowhere vanishing tangent vector field on M.
This shows that on an odd dimensional sphere, there always exists nan-vanishing (equivalently unit) tangent vector fields.
This is false for the 2 dimensional sphere S^2 but we won't prove this fact in this course.

16. Let V be a 3-dimensional inner product space. Fix two elements v, w in V.

(a) Then prove that there exists a unique vector $g(v, w)$ in V such that for all z in V, the following equation holds:

$$\langle g(v, w), z\rangle = \det(v, w, z)^t.$$

(**Hint:** Look at the map

$$\phi : V \to \mathbb{R},\ \phi(z) = \det(v, w, z)^t.$$

Observe that ϕ is a linear functional on V.)

(b) Prove that $g(v, w)$ coincides with the cross-product $v \times w$. From now on, we will drop the symbol $g(v, w)$ and instead continue to denote it as $v \times w$.

(c) From the above-made definition of $v \times w$, prove that $\det(v, w, v \times w)^t$ is always non-negative. Moreover, prove that $v \times w$ is orthogonal to both v and w.

(d) Prove that if (U, ψ) is a local parametrization of a 2-manifold in $\mathbb{R}^3$ such that U is a region, then there exists a unit normal vector field along ψ.

(e) Compute this unit normal vector field for the parametrization (U, ψ) where $U = \{(\theta, \phi) \in \mathbb{R}^2 : -\pi < \theta < \pi, 0 < \phi < \pi\}$ and $\psi : U \to \mathbb{R}^3$ is defined as

$$\psi(\theta, \phi) = (r \cos\theta \sin\phi, r \sin\theta \sin\phi, r \cos\phi).$$

17. Let (U, ψ) be a parametrized n-surface in $\mathbb{R}^{n+1}$. Let $X_1, X_2, \cdots X_n$ be the co-ordinate vector fields along ψ. Suppose $x \in U$. Prove that there is a unique vector $N(x) \in (\mathrm{Ran}(D\psi(x)))^{\perp}$ satisfying the following two conditions:

(a)

$$||N(x)|| := (< N(x), N(x) >_{T_{\psi(x)}(\mathbb{R}^{n+1})})^{\frac{1}{2}} = 1.$$

(b) The determinant of the matrix with the rows $X_1(x), X_2(x), \cdots N(x)$ (in this particular order) is positive.

18. The goal of this exercise is to show that the vector field N constructed in the previous problem is indeed a smooth vector field.

We continue with the notation of the previous exercise. Define

$$N'(x) = \sum_{i=1}^{n+1} n'_i(x) \frac{\partial}{\partial y_i}|_{\psi(x)},$$

where $n'_i(x) = (-1)^{n+i+1}$ times the determinant of the matrix obtained by deleting the i-th column from the $n \times (n+1)$ matrix with the first row $X_1(x)$, second row $X_2(x), \cdots$ the n-th row $X_n(x)$.

Here, the entries of the vector $X_i(x)$ are in the basis $\frac{\partial}{\partial y_i}|_{\psi(x)}$, where, $y_1, \cdots y_{n+1}$ are the co-ordinates of $\mathbb{R}^{n+1}$.

Prove that

(a) $N'(x) \neq 0$ for all $x \in U$.

(b) $N'(x) \in (\text{Ran}(D\psi(x)))^{\perp}$.

(c) The determinant of the matrix with the rows $X_1(x), X_2(x), \cdots X_n(x), N'(x)$ (in this particular order) is positive.

(d) Prove that the assignment $x \to N(x)$ constructed in the previous problem is a (smooth) vector field. The vector field N **is called the orientation vector field along** ψ.

(e) Now here comes the moral of the story.

Prove that if (U, ψ) is a local parametrization of an n-manifold in $\mathbb{R}^{n+1}$ so that U is a region, then there exists a unit normal vector field along ψ.