Assignment 7

1. Prove that $A \subseteq \mathbb{R}^n$ has content zero if and only if given $\epsilon > 0$, there exists a finite cover $\{U_1, \cdots U_n\}$ of A by open rectangles such that $\sum_{i=1}^{n} \text{vol}(\overline{U_i}) < \epsilon$.

2. Prove that the following subsets have measure zero:

   (a) any countable set in $\mathbb{R}^n$.

   (b) B, where $B \subseteq A$ and A has measure zero.

3. Prove that the following subsets do not have measure zero:

   (a) A, where A contains a set which does not have measure zero.

   (b) Suppose A is a subset of $\mathbb{R}^n$ which has one interior point, then A does not have measure zero.

4. Prove that if K is a compact set in $\mathbb{R}^n$ which has measure zero, then K has content zero.

5. (a) Suppose Ω_1 and Ω_2 are regions in $\mathbb{R}^n$ with $\Omega_1 \subseteq \Omega_2$. Prove that $\text{vol}(\Omega_1) \leq \text{vol}(\Omega_2)$.

   (b) Suppose $S \subseteq \mathbb{R}^n$ is a region such that $S \subseteq B_2(x_0, r)$ for some x_0 in $\mathbb{R}^n$, where $B_2(x_0, r)$ denotes the open ball around x_0 of radius r. Then prove that $\text{vol}(S) \leq 2^n r^n$.

6. Suppose f is a real-valued function which is continuous at a and integrable on a neighborhood of a, prove that

$$f(a) = \lim_{\epsilon \to 0^+} \frac{1}{\text{vol}(B(a, \epsilon))} \int_{B(a,\epsilon)} f(x_1, \cdots x_n) dx_1 \cdots dx_n.$$

7. Let ϕ_1, ϕ_2 be two continuous non-negative functions defined on $[a, b]$ such that $\phi_1(x) \leq \phi_2(x)$ for all x in $[a, b]$. Let S be the subset of $\mathbb{R}^2$ defined as

$$S = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)\}.$$

   Prove that if all iterated integrals of f exist, then

$$\int_S f(x, y) dx dy = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x, y) dy \right) dx.$$

8. Prove that any compact regular k-level surface in $\mathbb{R}^{n+k}$ has $(n + k)$-dimensional content zero.

9. This exercise shows that none of the hypotheses of Fubini's theorem can be dropped. We will have three cases.

   (a) Consider the function

   $$f : [0,1] \times [0,1] \to \mathbb{R},\ \ f(x,y) = 1 \text{ if } x = 0,\ \ y \in \mathbb{Q},\ \ 0 \text{ otherwise.}$$

   Then show that f is integrable on $[0,1] \times [0,1]$ but $\int_0^1 f(0,y)dy$ does not exist.

   (b) Consider the function

   $$f : [0,1] \times [0,1] \to \mathbb{R},\ \ f(x,y) = 1 \text{ if } y \in \mathbb{Q},\ \ 2x \text{ otherwise.}$$

   Then show that f is not integrable on $[0,1] \times [0,1]$ but the integrals $\int_0^1 f(x,y)dx$ and $\int_0^1 (\int_0^1 f(x,y)dx)dy$ exist.

   (c) Let q be a prime number. Consider the function

   $$f : [0,1] \times [0,1] \to \mathbb{R},\ f(x,y) = 1 \text{ if } x = \frac{m}{q},\ y = \frac{n}{q} \text{ for some } m,n \in \mathbb{N},\ 0 \text{ otherwise.}$$

   Then show that f is not integrable on $[0,1] \times [0,1]$ but $\int_0^1 \int_0^1 f(x,y)dxdy$ as well as $\int_0^1 \int_0^1 f(x,y)dydx$ exist.

10. Let $S = \{(x,y) : x^2 + y^2 \leq a^2, y \geq 0\}$. Evaluate $\int_S ydxdy$.

11. Let Ω be the subset of $\{(x,y,z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$ which is bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$. Evaluate $\int_\Omega xdxdydz$.

12. (*) The change of variable formula works if the derivative of the change of variable map g is invertible at all points of the domain of g. The goal of this exercise is to point out that this condition can be dropped under some circumstances.

    More precisely, the following statement is true:

    If $U \subseteq \mathbb{R}^n$ is an open region and $g : U \to \mathbb{R}^n$ is a one-one C^1-function such that the set

    $$B = \{x \in U : \det(Dg(x)) = 0\}$$

    is a region. Suppose in addition, the following conditions hold:

    (a) g extends to a C^1-function on an open set V containing $\overline{U}$.

    (b) $g(U)$ is a region.

    (c) $f : g(U) \to \mathbb{R}$ is a function which is Riemann-integrable on $g(U)$.

    (d) $f \circ g.\|\det(Dg)\|$ is Riemann-integrable on U.

Then

$$\int_{g(U)} f(x_1, \cdots x_n) dx_1 \cdots dx_n = \int_U f \circ g(y_1, \cdots y_n) |\det(Dg)(y_1, \cdots y_n)| dy_1 \cdots dy_n.$$

Now here comes the exercise:

Prove the above statement using the change of variable formula stated during the lecture and the Sard's theorem which states:

If $U \subseteq \mathbb{R}^n$ is an open set and $g : U \to \mathbb{R}^n$ is a C^1-function. Let

$$B = \{x \in U : \det(Dg(x)) = 0\}.$$

Then $g(B)$ has n-dimensional measure zero.

If you are interesting in the proof of Sard's theorem, have a look at Spivak, page 72, Theorem 3.14.

13. Let Ω be the subset of $\{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$ which is bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$. Evaluate $\int_\Omega x dx dy dz$ by using cylindrical co-ordinates.

14. Compute the volume of the closed 3-dimensional ball of radius r centered at the origin using spherical change of co-ordinates.

15. Suppose M is a k manifold in $\mathbb{R}^n$ and $p \in M$. Then prove that p belongs to the image of some parametrized k-surface in $\mathbb{R}^n$. Observe that this implies that any manifold can be covered by images of parametrized k-surfaces in $\mathbb{R}^n$.

16. This exercise gives some examples of parametrized n-surfaces in $\mathbb{R}^{n+1}$. In each case, prove that the example is indeed a parametrized surface. Moreover, compute the coordinate vector fields and the orientation vector fields along the parametrization. Finally, compute the volume of the parametrized surface.

    (a) Let a, b be two real numbers such that $a < b$. Define $h : [0, 1] \to \mathbb{R}^2$ by $h(t) = ((1 - t)b + ta, 0)$. Note that $h(0) = b$ and $h(1) = a$.

    (b) Let a, b be two real numbers such that $a < b$. Now define $h : [0, 1] \to \mathbb{R}^2$ by $h(t) = ((1 - t)a + tb, 0)$. Here, $h(0) = a$ and $h(1) = b$.

    Thus, the same set can have more than one parametrizations. Also note that in this new parametrization, the orientation vector field points in the opposite direction to the orientation vector field in part a.

    (c) Consider the function

    $$g : (0, \frac{\pi}{2}) \to \mathbb{R}^2$$

    defined as $g(x) = (\cos x, \sin x)$.

(d) Consider the function

$$g : \{(r,\theta) : 0 < r < R, 0 < \theta < \frac{\pi}{2}\} \to \mathbb{R}^3$$

is defined as $g(r,\theta) = (r\cos\theta, r\sin\theta, 0)$.

(e) Let

$$\Omega = \{(\theta, z) : 0 < \theta < \frac{\pi}{2}, \quad -M < z < M\}$$

and let $\psi : \Omega \to \mathbb{R}^3$ by

$$\psi(\theta, z) = (R\cos\theta, R\sin\theta, z)$$

for some $R > 0$.

(f) Let

$$\Omega = \{(r,\theta,\phi) : 0 < r < R, 0 < \theta < \frac{\pi}{2}, \ 0 < \phi < \frac{\pi}{2}\}$$

for some $R > 0$ and define $\psi : \Omega \to \mathbb{R}^4$ by

$$\psi(r,\theta,\phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi, 0).$$

17. Suppose f is a real valued smooth function on an open region U in $\mathbb{R}^n$. If $\phi : U \to \mathbb{R}^{n+1}$ is defined as

$$\phi(u_1, u_2, \cdots u_n) = (u_1, \cdots u_n, f(u_1, \cdots u_n)),$$

then prove that (U, ϕ) is a parametrized n-surface in $\mathbb{R}^{n+1}$.

18. Suppose $\gamma : (a,b) \to \mathbb{R}^2$ be a parametrized 1-surface in $\mathbb{R}^2$. Prove that

$$\text{vol}(\gamma(a,b)) = \int_a^b ||\dot{\gamma}(t)||dt.$$

19. Let $U = \{(\theta,\phi) \in \mathbb{R}^2 : 0 < \theta < \frac{\pi}{2}, 0 < \phi < \frac{\pi}{2}\}$ and $\psi : U \to \mathbb{R}^3$ be defined as

$$\psi(\theta,\phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi).$$

(a) Compute the volume of this parametrized surface.

(b) Does the value of the integral remind you of something? What is the image of ψ?

20. Let (U,ψ) be a parametrized n-surface in $\mathbb{R}^{n+1}$ and $X_1, \cdots X_n$ be the co-ordinate vector fields along ψ while N will denote the orientation vector field along ψ.

For $i, j = 1, 2, \cdots n$, $g_{ij} : U \to \mathbb{R}$ be the functions defined as

$$g_{ij}(u_1, \cdots u_n) =< X_i(u_1, \cdots u_n), X_j(u_1, \cdots u_n) > .$$

Define $g : U \to M_n(\mathbb{R})$ by the formula

$$g(u_1, \cdots u_n) = (g_{ij}(u_1, \cdots u_n))_{ij}.$$

Prove that

$$\mathrm{vol}(\psi(U)) = \int_U (\det g(u_1, \cdots u_n))^{\frac{1}{2}} du_1 \cdots du_n.$$

(**Hint:** We indicate the hint for a parametrized 2-surface in $\mathbb{R}^3$. The general case follows in the same way.

$$(\det \begin{pmatrix} X_1(p) \\ X_2(p) \\ N(p) \end{pmatrix})^2 = \det \begin{pmatrix} X_1(p) \\ X_2(p) \\ N(p) \end{pmatrix} \times \det \begin{pmatrix} X_1(p) \\ X_2(p) \\ N(p) \end{pmatrix}^t$$

$$= \det(\begin{pmatrix} X_1(p) \\ X_2(p) \\ N(p) \end{pmatrix} \times \big((X_1(p))^t \quad (X_2(p))^t \quad (N(p))^t\big))$$

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