Assignment 8

1. If V is a vector space of dimension n, then prove that $\Lambda^k(V) = 0$ if $k > n$.

2. (a) Prove that if ω is a k-form on an open subset U, k being odd, then $\omega \wedge \omega = 0$.

   (b) Suppose the coordinates in $\mathbb{R}^4$ are given by x_1, x_2, y_1, y_2. Consider the 2 form in $\mathbb{R}^4$ defined by
   $$\omega = d(x_1) \wedge d(y_1) + d(x_2) \wedge d(y_2).$$
   Then prove that
   $$\omega \wedge \omega = 2d(x_1) \wedge d(y_1) \wedge d(x_2) \wedge d(y_2)$$
   and hence $\omega \wedge \omega \neq 0$.

3. Compute the exterior derivative of the following differential forms:

   (a) $\omega = e^{xy}dx$ considered as a one-form in $\mathbb{R}^2$.

   (b) $\omega = z^2dx + x^2dy + y^2dz$ considered as a one form in $\mathbb{R}^3$.

   (c) $\omega = x_1x_2dx_3 \wedge dx_4$ considered as a two-form in $\mathbb{R}^4$.

4. Compute the pullback $g^*\omega$ for the following examples:

   (a) $g(u, v) = (\cos u, \sin u, v)$ and $\omega = zdx + xdy + ydz$.

   (b) g being the spherical co-ordinate map from $(0, \infty) \times (0, 2\pi) \times (0, \pi)$ to $\mathbb{R}^3$ and $\omega = dx \wedge dy \wedge dz$.

5. Suppose U, V, W are open sets in $\mathbb{R}^n, \mathbb{R}^m$ and $\mathbb{R}^p$ respectively. If $f : U \to V$ and $g : V \to W$ are smooth functions, then prove that
   $$(g \circ f)^*\omega = (f^* \circ g^*)\omega.$$

6. Prove that if U is an open subset of $\mathbb{R}^n$, then $dx_1 \wedge dx_2 \wedge \cdots dx_n$ is a nowhere vanishing form on U.

7. Let U be an open set in $\mathbb{R}^n$. Suppose $\omega : U \to \cup_{q \in U} \Lambda^k(T_q(U))$ is a map such that $\omega(p) \in \Lambda^k(T_p(U))$ for all $p \in U$. Prove that $\omega \in \Omega^k(U)$ if and only if for all $X_1, X_2, \cdots X_k \in \mathfrak{X}(U)$, the map
   $$\omega_{X_1, \cdots X_k} : U \to \mathbb{R},\ \omega_{X_1, \cdots X_k}(x) = \omega(x)(X_1(x), X_2(x), \cdots X_k(x))$$
   is C^∞.

8. Suppose $\gamma : [a, b] \to \mathbb{R}^n$ defines a parametrized 1-surface in $\mathbb{R}^n$ and let $\gamma = (\gamma_1, \cdots \gamma_n)$. Let $\omega = \sum_{i=1}^{n} f_i dx_i$ is a one-form on $\mathbb{R}^n$.

   (a) Prove that $\gamma^*(dx_i) = \gamma_i^{\cdot} dt$, where dt denotes the generating one-form on $\mathbb{R}$.

   (b) Prove that

$$\int_{\gamma([a,b])} \omega = \sum_{i=1}^{n} \int_a^b (f_i \circ \gamma)(t)\gamma_i^{\cdot}(t)dt.$$

9. Let C be the line segment joining $(1, -1, 0)$ and $(2, 2, 2)$ in $\mathbb{R}^3$ and let $\omega = xydz$. Give a suitable parametrization of C and calculate $\int_C \omega$.

10. Consider the rectangle $R = [a, b] \times [c, d]$. Endow ∂R with the anti-clockwise parametrization, i.e,

$$\begin{aligned} \gamma(t) &= \gamma_1(t),\ 0 \le t < 1 \\ &= \gamma_2(t),\ 1 \le t < 2 \\ &= \gamma_3(t),\ 2 \le t < 3 \\ &= \gamma_4(t),\ 3 \le t < 4, \end{aligned}$$

where

$$\begin{aligned} \gamma_1(t) &= ((1-t)a + tb, c) \\ \gamma_2(t) &= (b, (2-t)c + (t-1)d) \\ \gamma_3(t) &= ((3-t)b + (t-2)a, d) \\ \gamma_4(t) &= (a, (4-t)d + (t-3)c). \end{aligned}$$

   (a) Compute $\int_{\partial R} f dx + g dy$.

   (b) Compute $\int_R d\omega$.

   (c) Prove the Green's theorem for rectangles:
   Let $R \subseteq \mathbb{R}^2$ be a 2-dimensional rectangle and let $\omega \in \Omega^1(U)$, where U is an open set in $\mathbb{R}^2$ containing R. -Then

$$\int_{\partial R} \omega = \int_R d\omega,$$

   where ∂R is given the anticlockwise parametrization.

   (d) Prove that the Green's theorem fails if the boundary ∂R is given a clockwise parametrization.

11. Consider the trapezium with vertices $(a, 0), (b, 0), (e, f), (c, d)$. . Here, $b > a$, $e > a$, $f > 0$, $c < b$ and $d > 0$.

    Moreover, let $R = [0, 1] \times [0, 1]$.

(a) Prove that the following equations define a parametrization

$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 : \partial R \to \mathbb{R}^2$$

of the boundary of the trapezium.

$$\begin{aligned}
\gamma_1(t,0) &= (1-t)(a,0) + t(b,0) \\
\gamma_2(1,t) &= (1-t)(b,0) + t(c,d) \\
\gamma_3(1-t,1) &= (1-t)(c,d) + t(e,f) \\
\gamma_4(0,1-t) &= (1-t)(e,f) + t(a,0).
\end{aligned}$$

(b) Prove that the interior of R parametrizes the interior of the trapezium by the equation

$$\psi(x,y) = (1-x)\gamma_4(0,y) + x\gamma_2(1,y).$$

(c) Using the Green's theorem for the rectangle, prove the Green's theorem for the trapezium.

(d) Prove Green's theorem for the closed half-disk $\{(x,y) \in \mathbb{R}^2 : x \in [-1,1], 0 \leq y \leq \sqrt{1-x^2}\}$.

12. (Gradient, divergence and curl)

Suppose U is an open set in $\mathbb{R}^3$.

(a) If $X = \sum_{i=1}^3 f_i \frac{\partial}{\partial x_i}$ is a vector field, then the work form associated to X is the one-form W_X on U defined by

$$W_X(p)(v) = \langle X_p, v \rangle,$$

where $v \in T_pU$ and the inner product is taken in the vector space $T_p(U)$.
Prove that if $X = \sum_{i=1}^3 f_i \frac{\partial}{\partial x_i}$, then

$$W_X = \sum_{i=1}^3 f_i dx_i.$$

(b) The flux form Φ_X associated to a vector field X on U is the two-form on U defined by

$$\Phi_X(p)(v,w) = \det(X_p, v, w)^t$$

for all $v, w \in T_p(U)$.
Here, $(X_p, v, w)^t$ is the transpose of the matrix (X_p, v, w).
Prove that if if $X = \sum_{i=1}^3 f_i \frac{\partial}{\partial x_i}$, then

$$\Phi_X = f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy.$$

(c) If $f \in C^\infty(U)$ (i.e, f is a scalar field), then the mass form M_f is the three-form defined by

$$M_f(p)(v_1, v_2, v_3) = f(p)\det(v_1, v_2, v_3)^t$$

for all v_1, v_2, v_3 in $T_p(U)$.

Prove that $M_f = f dx \wedge dy \wedge dz$.

(d) Suppose $X, Y \in \mathfrak{X}(U)$, then prove the following equations:

i. Let $X \times Y$ be the vector field on U defined by

$$(X \times Y)(p) = X_p \times Y_p,$$

where $\times$ denotes the cross-product of two vectors in $\mathbb{R}^3$.

Prove that

$$\Phi_{X \times Y} = W_X \wedge W_Y.$$

ii. Let $X \cdot Y$ be scalar field on U defined by

$$(X.Y)(p) = \langle X_p, Y_p \rangle,$$

where the inner product has been taken in the vector space $T_p(U)$.

Prove that

$$M_{X \cdot Y} = W_X \wedge \Phi_Y = W_Y \wedge \Phi_X.$$

(e) Now let us recall the definitions of gradient, curl and divergence.

i. The gradient of a scalar field f is defined to be the vector field

$$\nabla f = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}.$$

ii. The curl of a vector field $X = \sum^3 f_i \frac{\partial}{\partial x_i}$ is defined to be the vector field

$$\nabla \times X = (\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3})\frac{\partial}{\partial x_1} + (\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1})\frac{\partial}{\partial x_2} + (\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2})\frac{\partial}{\partial x_3}.$$

iii. The divergence of a vector-field $X == \sum^3 f_i \frac{\partial}{\partial x_i}$, denoted by $\mathrm{div}(X)$ is the scalar field on U defined by

$$\nabla . X = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}.$$

Now, for $f \in C^\infty(U)$ (i.e, a scalar field on U) and a vector field X on U, prove that

$$df = W_{\nabla f}, \; dW_X = \Phi_{\nabla \times X}, \; d\Phi_X = M_{\nabla . X}.$$

Observe that these three equations taken together prove that the diagram in the attached file is commutative.

(f) Using the commutativity of the above diagram and the relation $d^2 = 0$, prove that

$$\nabla \times \nabla f = 0 = \text{div}(\nabla \times X).$$

(g) If f is a scalar field on U, then the Laplacian of f is defined as $\Delta f := \text{div}\nabla f$. Prove that

$$\Delta f = \sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2}.$$

13. Suppose (U, ψ) is a parametrized n-surface in $\mathbb{R}^{n+1}$, where U is an open region and let N be the orientation vector field along ψ introduced in Assignment 6. Recall that we defined $\text{Vol}(\psi(U))$ to be the quantity

$$\int_U \det(X_1(u_1, \cdots u_n), X_2(u_1, \cdots u_n), \cdots, X_n(u_1, \cdots u_n), N(u_1, \cdots u_n))^t du_1 \cdots du_n. \tag{1}$$

(a) Prove that $\text{Vol}(\psi(U))$ is positive.

(b) Prove that

$$\text{Vol}(\psi(U)) = \int_U [\det(g(u_1, \cdots u_n))]^{\frac{1}{2}} du_1 \cdots du_n,$$

where $g(u_1, \cdots u_n)$ is the $M_n(\mathbb{R})$-valued function on U whose (i, j)-th entry is $\langle X_i(u_1, \cdots u_n),\ X_j(u_1, \cdots u_n)\rangle$.

(c) Suppose (U, ψ) is a local parametrization of an oriented n-manifold (M, ω) in $\mathbb{R}^{n+1}$ where U is an open region. Observe that $\psi(U)$ is also a manifold. Prove that $\text{Vol}(\psi(U))$ as defined by equation (1) is equal to $\int_U \psi^*(\text{dvol}_M)$ if (U, ψ) is positively oriented. Thus, the two definitions of volume agree on the manifold $\psi(U)$.

(d) In Assignment 7, we computed $\text{vol}(\psi(U))$, where $U = \{(\theta, \phi) \in \mathbb{R}^2 : 0 < \theta < \frac{\pi}{2}, 0 < \phi < \frac{\pi}{2}\}$ and $\psi : U \to \mathbb{R}^3$ be defined as

$$\psi(\theta, \phi) = (r \cos\theta \sin\phi, r \sin\theta \sin\phi, r \cos\phi).$$

Now here is a follow up problem:
Construct an orientation form η on S^2 such that $\text{vol}(\psi(U))$ is the volume of the manifold $(\psi(U), \eta)$.

(e) Suppose $\gamma : [a, b] \to \mathbb{R}^2$ define a parametrized 1-surface. Prove that

$$\text{vol}(\gamma([a, b]) = \int_a^b ||\gamma^{\cdot}(t)||dt.$$

14. Prove that a regular n-level surface in $\mathbb{R}^{n+k}$ is orientable. compute the volume form corresponding to the orientation form you have constructed.

15. Suppose f is a real valued smooth function on an open set U in $\mathbb{R}^n$. If $\phi : U \to \mathbb{R}^{n+1}$ is defined as

$$\phi(u_1, u_2, \cdots u_n) = (u_1, \cdots u_n, f(u_1, \cdots u_n)),$$

then prove that

 (a) $(U, \phi, \text{Graph}(f))$ is a parametrized n-surface in $\mathbb{R}^{n+1}$.

 (b) Show that the orientation vector field along ϕ is given by

$$N = \frac{(-\frac{\partial f}{\partial u_1}, \cdots, -\frac{\partial f}{\partial u_n}, 1)}{[1 + \sum_{i=1}^{n} (\frac{\partial f}{\partial u_i})^2]^{\frac{1}{2}}}.$$

 (c) Compute the volume of Graph(f).

16. Let (U, ψ) be a parametrized 2-surface in $\mathbb{R}^3$ and let X_1, X_2 denote the coordinate vector fields along ψ. We define three functions E, F, G on U as

$$E = \langle X_1, X_1 \rangle, G = \langle X_2, X_2 \rangle, F = \langle X_1, X_2 \rangle,$$

i.e, for $p \in U$, $E(p) = \langle X_1(p), X_1(p) \rangle_{T_{\psi(p)}(\psi(U))}$, etc.

Then prove that

$$\text{Vol}(\psi(U)) = \int_U \sqrt{EG(u_1, u_2) - F^2(u_1, u_2)} du_1 du_2.$$

17. If S is a regular n-level surface with boundary in $\mathbb{R}^{n+1}$, then prove that $\partial_M S$ is a disjoint union of regular $n-1$ level surfaces in $\mathbb{R}^{n+1}$.

18. Consider the following subsets of Euclidean spaces:

 (a) The closed unit disk in $\mathbb{R}^2$.

 (b) The set $\overline{B(a, r)} := \{x \in \mathbb{R}^n : \|x - a\| \leq r\}$.

 (c) The closed annulus in $\mathbb{R}^2$, i.e, the set $\{(x, y) \in \mathbb{R}^2 : a \leq x^2 + y^2 \leq b\}$, where a, b are two positive real numbers.

Then show that all these subsets have the following property (for a certain choice of n in each of the cases), which we shall call **Property** $*$ for the moment:

S is a compact regular n-surface with boundary in $\mathbb{R}^{n+1}$ of the form $f^{-1}(0) \cap (\cap_{i=1}^{k} g_i^{-1}(-\infty, c_i])$ with $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $f(x_1, \cdots, x_{n+1}) = x_{n+1}$.

Note that if S satisfies property $*$, then $S \subseteq \mathbb{R}^n \times \{0\}$.

In each of the above mentioned examples, identify the manifold boundaries.

19. The following observations are needed in the proof of the divergence theorem:

(a) Suppose V is a vector space of dimension n and $\{e_1, \cdots e_n\}$ is an orthonormal basis of V. If $X, Y \in \Lambda^n(V)$ are such that $X(e_1, \cdots e_n) = Y(e_1, \cdots e_n)$, then prove that $X = Y$ as elements of $\Lambda^n(V)$.

(b) Suppose M is a compact k-manifold in $\mathbb{R}^n$ and ω, η are k-forms on M.

Recall that this means that there exists an open set W in $\mathbb{R}^n$ which contains M and that $\omega, \eta \in \Omega^k(W)$.

Suppose for all $x \in M$ and for all $\{v_1, \cdots v_n\}$ in T_xM, we have

$$\omega(x)(v_1, \cdots v_n) = \eta(x)(v_1, \cdots v_n).$$

Prove that $\int_M \omega = \int_M \eta$.

(c) If S has the property $*$ as in the previous problem, and X is a vector field defined on an open subset V of $\mathbb{R}^n$ containing S, then prove that X can be extended to a smooth vector field on the set $V \times \mathbb{R}$ which is an open set in $\mathbb{R}^{n+1}$.

(d) Suppose S has the property $*$ as in the previous problem. If $x_1, \cdots x_n, x_{n+1}$ denotes the co-ordinates on $\mathbb{R}^{n+1}$ and the orientation form on $\mathbb{R}^n$ is defined to be $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$, then prove that

$$\mathrm{dvol}_S = dx_1 \wedge dx_2 \cdots dx_n.$$

(e) Suppose S has the property $*$ as in the previous problem so that we have $\mathrm{dvol}_S = dx_1 \wedge dx_2 \cdots \wedge dx_n$. Prove that

$$i_{f_j \frac{\partial}{\partial x_j}}(\mathrm{dvol}_S) = (-1)^j f_j dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

where the symbol $\widehat{dx_j}$ means that dx_j is not present in the term.

20. (a) Let V be a 3-dimensional inner product space. Fix two elements v, w in V. Then prove that there exists a unique vector $g(v, w)$ in V such that for all z in V, the following equation holds:

$$\langle g(v, w), z \rangle = \det(v, w, z)^t.$$

(**Hint:** Look at the map

$$\phi : V \to \mathbb{R}, \ \phi(z) = \det(v, w, z)^t.$$

Observe that ϕ is a linear functional on V.)

(b) Prove that $g(v, w)$ coincides with the cross-product $v \times w$. From now on, we will drop the symbol $g(v, w)$ and instead continue to denote it as $v \times w$.

(c) From the above-made definition of $v \times w$, prove that $\det(v, w, v \times w)^t$ is always non-negative. Moreover, prove that $v \times w$ is orthogonal to both v and w.

(d) Now suppose that S is a compact connected regular 2-level surface in $\mathbb{R}^3$ and let n be a nowhere vanishing normal vector field on S. If we orient S by the vector field n, then prove that for all $x \in S$ and for all $v, w \in T_x S$,

$$\text{dvol}(x)(v, w) = \det(v, w, n(x))^t. \tag{2}$$

(e) Let S be as above. Prove that for all x in S and for all $v, w \in T_x S$ and for all $z \in T_x \mathbb{R}^3$, the following equation holds:

$$\langle z, n(x) \rangle \, \text{dvol}_S(x)(v, w) = \langle z, v \times w \rangle \, . \tag{3}$$

(**Hint:** Use the equation (2). Remember that $v \times w$ is a scalar multiple of $n(x)$.)

21. Let S be a compact connected regular 2-level surface with boundary in $\mathbb{R}^3$. Let

$$n = n_1 \frac{\partial}{\partial x} + n_2 \frac{\partial}{\partial y} + n_3 \frac{\partial}{\partial z}$$

be a unit normal orientation vector field on S.

(a) Prove that the volume form (should be called the area-form in this case) is given by

$$\text{dvol}_S = n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy.$$

(**Hint:** Let $\omega = n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy$. Observe that it is enough to prove that if $(v, w) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ or $(\frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ or $(\frac{\partial}{\partial z}, \frac{\partial}{\partial x})$, then

$$\omega(x)(v, w) = \text{dvol}_S(x)(v, w).$$

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(b) Moreover, prove that the following equations hold:

$$n_1 \text{dvol}_S = dy \wedge dz, \; n_2 \text{dvol}_S = dz \wedge dx, \; n_3 \text{dvol}_S = dx \wedge dy. \tag{4}$$

(**Hint:** Use (3) with suitable choices of z. Remember that $dx \wedge dy(v, w)$ is the determinant of a 2×2 minor of a 2×3 matrix.)

22. Tho goal of this exercise is to derive the classical version of the Stokes' formula from the version of the Stokes theorem presented during the lecture.

Let S be a compact connected oriented regular level 2-surface with boundary in $\mathbb{R}^3$. Let X be a vector field on an open set V' in $\mathbb{R}^3$ such that $S \subseteq V'$. Let $\nabla \times X$ denote $\text{curl}(X)$, N the unit normal vector on S consistent with the orientation and T be the unique tangent vector field on $\partial_M S$ with $\text{dvol}_{\partial_M S}(T) = 1$.

Then the classical Stokes' formula states that

$$\int_S \langle \nabla \times X, N\rangle \,\mathrm{dvol}_S = \int_{\partial_M S} \langle X, T\rangle \,\mathrm{dvol}_{\partial_M S}. \tag{5}$$

The classical Stokes' formula follows by applying the Stokes' theorem to the work-form W_X associated to the vector field X.

(a) Prove that

$$\int_S dW_X = \int_S \langle \nabla \times X, N\rangle \,\mathrm{dvol}_S.$$

(**Hint:** Use the equation (4) from the previous problem.)

(b) Prove that

$$\int_{\partial_M S} W_X = \int_{\partial_M S} \langle X, T\rangle \,\mathrm{dvol}_{\partial_M S}.$$

(**Hint:** Remember that $\partial_M S$ is a one-manifold. If (U, γ) is a positively oriented local parametrization of $\partial_M S$, then it is enough to prove that for all smooth function f such that $0 \leq f \leq 1$, we have

$$\int_U f \circ \gamma.\gamma^*(W_X) = \int_U f \circ \gamma \,\langle X, T\rangle \circ \gamma.\gamma^*(\mathrm{dvol}_{\partial_M S}).$$

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(c) Now combine the above two statements along with the Stokes theorem to derive the classical Stokes' formula (5).

23. Compute the flux of the vector field

$$X = xz^2 \frac{\partial}{\partial x} + yx^2 \frac{\partial}{\partial y} + zy^2 \frac{\partial}{\partial z}$$

outward across the surface $x^2 + y^2 + z^2 = a^2$.

You can use the usual spherical co-ordinate parametrization $\psi : (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$ is defined by

$$\psi(\phi, \theta) = (a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi).$$

24. Consider the two-form ω on $\mathbb{R}^3$ defined by:

$$\omega = xz dy \wedge dz + yz dz \wedge dx + (x^2 + y^2) dx \wedge dy.$$

We define a subset Ω of the paraboloid $z = 4 - x^2 - y^2$ as follows:

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : z = 4 - x^2 - y^2, z \geq 0\}.$$

We declare the orientation on Ω to be the one which corresponds to the normal vector field $2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. Compute $\int_\Omega \omega$.