Some problems on IMT and IFT

1. If U and V are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and $\phi : U \to V$ is a diffeomorphism, prove that $m = n$. This says that if $m \neq n$, then an open subset of $\mathbb{R}^n$ cannot be diffeomorphic to an open subset of $\mathbb{R}^n$.

2. Define $f\colon \mathbb{R}^5 \to \mathbb{R}^2$, given by $f = (f_1, f_2)$ where $f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1} + x_2y_1 - 4y_2 + 3$ and $f_2 = (x_1, x_2, y_1, y_2, y_3) = x_2 cos(x_1) - 6x_1 + 2y_1 - y_3$.
   (a) Show that $f(0, 1, 3, 2, 7) = (0, 0)$
   (b) Show that $\exists$ a C^1 map g defined on a neighbourhood of $(3, 2, 7)$ such that $g(3, 2, 7) = (0, 1)$ and $f(g(y), y) = (0, 0)$.
   (c) Compute $Dg(3, 2, 7)$.

3. Using the implicit function theorem (and not otherwise), show that the system of equations:
$$3x + y - z + u^2 = 0$$
$$x - y + 2z + u = 0$$
$$2x + 2y - 4z + 2u = 0$$
has a solution for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x.

4. Define $f\colon \mathbb{R}^3 \to \mathbb{R}$ by
$$f(x, y_1, y_2) = x^2y_1 + e^x + y_2$$
Show that $\frac{\partial f}{\partial x}(0, 1, -1) \neq 0$ and there exists a differentiable function g in a neighborhood of $(1, -1)$ in $\mathbb{R}^2$ so that $g(1, -1) = 0$ and $f(g(y_1, y_2), y_1, y_2) = 0$. Moreover find $\frac{\partial g}{\partial y_1}(1, -1)$ and $\frac{\partial g}{\partial y_2}(1, -1)$.