M. Math. Measure Theory
Problems and complements 1

1. Let A be the set of rational numbers between 0 and 1, and let $\{I_n\}$ be a finite collection of open intervals covering A. The show that $\sum l(I_n) \geq 1$.

2. Given any set A and any $\epsilon > 0$, show that there is an open set O such that $A \subset O$ and $m^\star(O) \leq m^\star(A) + \epsilon$. (**Sketch:** take $A \subset \cup I_j, \sum l(I_j) \leq m^\star(A) + \epsilon$. But clearly for $O = \cup I_j, m^\star(O) \leq \sum l(I_j)$, why? You may also increase each interval by $\epsilon/2^j$ and proceed. What if $m^\star(A) = \infty$?)

3. Prove that $m^\star$ is translation invariant.

4. Prove that if $m^\star(A) = 0$, then $m^\star(A \cup B) = m^\star(B)$.

5. Show that if E is a measurable set then each translate $E + y$ of E is also measurable.

6. Show that if E_1 and E_2 are measurable then $m(E_1 \cup E_2) + m(E_1 \cap E_1) = m(E_1) + m(E_2)$.

7. Give a decreasing sequence $\{E_n\}$ of measurable sets with $\phi = \bigcap E_n$ and $m(E_n) = \infty$ for all n.

8. Let $\{E_i\}$ be a sequence disjoint measurable sets and A any set. Then show that $m^\star(A \cap \cup_{i=1}^\infty E_i) = \sum_{i=1}^\infty m^\star(A \cap E_i)$.

9. Consider $[0, 1]$. Remove the middle third $(1/3, 2/3)$. Then from each of the remaining intervals remove the middle thirds, i.e. remove $(1/9, 2/9)$ and $(7/9, 8/9)$, and so on. The limiting set left is called the Cantor set C. The ternary expansion of any number in C has only 0 and 2 for its digits. (a) Uniqueness? Some rationals in this type of expansion (on entire $[0, 1]$) may have two expressions, e.g. .1 and $.022\ldots$. (b) Show that C can be put into a one to one correspondence with $[0, 1]$ so that its cardinality is c. (c) Is C a Borel set? (d) Show that C has measure zero. Thus there are uncountable sets with measure zero.

10. A set which is a countable union of closed sets is called an F_σ. A set which is a countable intersection of open sets is called a G_δ. Show that given any set A there is a $G \in G_\delta$ such that $A \subset G$ and $m^\star(G) = m^\star(A)$.

    Sketch: $\exists$ open $I_{n,i}$ such that $A \subset O_n = \cup_i I_{n,i}$ and $m^\star(O_n) \leq m^\star(A) + \epsilon_n$ where $\epsilon_n \downarrow 0$. Consider $G_n = \cap_{k=1}^n O_k \downarrow G$. $A \subset G$ and $m^\star(G) \leq m^\star(G_n) \leq m^\star(O_n) \leq m^\star(A) + \epsilon_n, \forall n$.

11. If E is measurable, then given $\epsilon > 0$ (i) exists open O such that $E \subset O$ and $m^\star(O \backslash E) \leq \epsilon$, (ii) exists closed F such that $F \subset E$ and $m^\star(E \backslash F) \leq \epsilon$, (iii) $m(E) < \infty$ implies exists compact K such that $K \subset E$ and $m(E \backslash K) \leq \epsilon$, (iv) $m(E) < \infty$ implies exists finite union of open (resp. closed) intervals U (resp. F) such that $m(E \Delta U)$ (resp. $m(E \Delta F)) \leq \epsilon$.

    Sketch: (i) Constructed O earlier, now E measurable implies $m^\star(O) = m^\star(O \cap E) + m^\star(O \cap E^c) = m^\star(E) + m^\star(O \backslash E)$. But $m^\star(O) \leq m^\star(E) + \epsilon$. (ii) E^c measurable, exists open O such that $E^c \subset O, m^\star(O \backslash E^c) < \epsilon$. Take $F = O^c$ and see that $E \backslash F = O \backslash E^c$. (iii) Pick F closed, $F \subset E, m^\star(E \backslash F) \leq \epsilon/2$. $[-n, n] \cap F = K_n$ are compact (why?) and $E \backslash K_n \downarrow E \backslash F, m(E) < \infty \Rightarrow \exists n \ni m(E \backslash K_n) \leq \epsilon$. (iv) $m(E) < \infty$. Choose $E \subset \cup I_j, \sum l(I_j) < m(E) + \epsilon/2$. Choose $N \ni \sum_{N+1}^\infty l(I_j) < \epsilon/2$. If $U = \cup_1^N I_j, m(E \Delta U) = m(E \backslash U) + m(U \backslash E) \leq m(\cup_{N+1}^\infty I_j) + m((\cup_1^\infty I_j) \backslash E) \leq \sum_{N+1}^\infty l(I_j) + \sum_1^\infty l(I_j) - m(E) \leq \epsilon/2 + \epsilon/2$. Verify the last but one inequality.

12. Show that (i) E measurable, implies (ii) given $\epsilon > 0$, exists open $O \supset E \ni m^\star(O \backslash E) < \epsilon$, implies (iii) exists $G \in G_\delta \ni E \subset G$ and $m^\star(G \backslash E) = 0$, implies (i). (In the last use that $m^\star(A) = 0$ implies A measurable, G_δ's are measurable and $E = G \backslash (G \backslash E)$.) Formulate a similar statement with F_σ and prove it.

    Problem 11, for $m^\star(A) = \infty$: $A_n = A \cap [n, n+1)$ can be enclosed in O_n with $m^\star(O_n \backslash A_n) < \epsilon_n$ so that $\sum \epsilon_n < \epsilon$. See that $\cup A_n = A, (\cup O_n) \backslash A = \cup (O_n \backslash A) \subset \cup (O_n \backslash A_n)$.

Remark: Problem 11 is referred to as approximation from above by open sets and from below by compact sets.

M. Math. Measure Theory
Problems and Complements 2

1. (a) Suppose A is a subset of $\boldsymbol{R}$. Describe the algebra generated by A. (b) If you start with two sets A, B then describe the algebra generated by them (i.e. smallest algebra containing A, B). (c) What if you started with $A_1, A_2, \cdots, A_k$ and wanted the smallest algebra containing them? Remember to include ϕ and $\boldsymbol{R}$ in these algebras.

2. Consider a function 1_E where E is any set. For such functions show that (a) $1_{A\cap B} = 1_A 1_B$, (b) $1_{A\cup B} = 1_A + 1_B - 1_A.1_B$, (c) $1_{A^c} = 1 - 1_A$. (d) $1_{A\Delta B} = |1_A - 1_B|$.

3. Let f be a function with measurable domain D. Show that f is measurable iff the function g defined by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

4. Suppose E is any set and consider the function $f = 1_E$. What are the sets $f^{-1}[a, \infty)$ for (a) $a \leq 0$, (b) $0 < a \leq 1$ (c) $a > 1$?

5. If f is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ then for open set O why is $f^{-1}(O)$ open? Is such an f measurable?

6. A step function with domain $\mathbb{R}$ is one which takes a finite number of distinct real values $c_1, c_2, \ldots, c_n$ (assume these are in increasing order) and $f^{-1}(c_i) = E_i$ is a union of a finite number of intervals (the intervals can be infinite in length). The E_i's are disjoint and their union is $\mathbb{R}$. Are step functions measurable?

7. A simple function with domain $\mathbb{R}$ is a measurable function which assumes a finite number of (distinct) real values. (a) Show that such a function can be written as $\sum_1^n c_i 1_{E_i}$ where c_i's are distinct real numbers and E_i's are disjoint measurable sets whose union is $\mathbb{R}$. (b) WLG assuming the c_i's are arranged in increasing order describe $f^{-1}[a, \infty)$ where $a \in \mathbb{R}$. (c) Just from the representation in (a) show that the sum and product of two simple functions is measurable.

8. Suppose f is a nonnegative function. Show that

$$\phi_k = \sum_{i=0}^{k2^k-1} \frac{i}{2^k} 1_{f^{-1}(\frac{i}{2^k}, \frac{i+1}{2^k}]} + k 1_{f^{-1}(k,\infty]},$$

is (take the first interval to be $[0, 1/2^k]$) a sequence of functions increasing to f everywhere. (At the $(k+1)$st level the measurable set $f^{-1}(\frac{i}{2^k}, \frac{i+1}{2^k}]$ gets decomposed into $f^{-1}(\frac{2i}{2^{k+1}}, \frac{2i+1}{2^{k+1}}] \cup f^{-1}(\frac{2i+1}{2^{k+1}}, \frac{2i+2}{2^{k+1}}]$ on which ϕ_{k+1} can take values $2i/2^{k+1}$ or $(2i+1)/2^{k+1}$ which are greater than or equal to the value $i/2^k$ of ϕ_k on the union. Also $f^{-1}((k, \infty]) = f^{-1}((k, k+1]) \cup f^{-1}(k+1, \infty])$, value of ϕ_k on the LHS is k, value of ϕ_{k+1} on $f^{-1}((k, k+1])$ is of the form $j/2^{k+1}, j = k.2^{k+1}$ to $(k+1)2^{k+1} - 1$, and $k+1$ on $f^{-1}(k+1, \infty])$. Thus $\phi_k \leq \phi_{k+1}$.

   (If you omit $k1_{f^{-1}(k,\infty]}$ then the limit of ϕ_k's gives only the real valued part of f.)

9. In the same set up if you define $F_k(x) = f(x)$ if $x \in [-k, k]$ or $f(x) \leq k$, $F_k(x) = k$ if $x \in [-k, k]$ and $f(x) > k$, and $F_k(x) = 0$ otherwise then show that F_k are nondecreasing. (either support increases and recall f is nonnegative, or F_{k+1} evaluated at the same point where F_k is positive may be larger.)

10. Thus F_{k+1} evaluated at points of $F_k^{-1}(\frac{i}{2^k}, \frac{i+1}{2^k}]$ is greater than or equal to the value of F_k at the same point. From this show that

$$\psi_k = \sum_{i=0}^{k2^k-1} \frac{i}{2^k} 1_{F_k^{-1}(\frac{i}{2^k}, \frac{i+1}{2^k}]},$$

are nondecreasing and go to f everywhere. See that this ψ_k requires simple functions of measurable sets of finite measure, in fact sets which are also bounded at each kth step.

11. (a) Given a measurable function f on $[a, b]$ that takes the values $\pm\infty$ only on a set of measure zero, and given $\epsilon > 0$, there is an M such that $|f| \leq M$ except on a set of measure less than $\epsilon/3$.

(b) Let f be a measurable function on $[a, b]$. Given $\epsilon > 0$ and M, there is a simple function ϕ such that $|f(x) - \phi(x)| < \epsilon$ except where $|f(x)| \geq M$. If $m \leq f \leq M$, then we may take ϕ so that $m \leq \phi \leq M$.

(c) Given a simple function ϕ on $[a, b]$, there is a step function g on $[a, b]$ such that $g(x) = \phi(x)$ except on a set of measure less than $\epsilon/3$. If $m \leq \phi \leq M$, then we can take g so that $m \leq g \leq M$. (Outside the set the exact value may not have any relation with ϕ.)

(d) Given a step function g on $[a, b]$, there is a continuous function h such that $g(x) = h(x)$ except on a set of measure less than $\epsilon/3$. If $m \leq g \leq M$, then we may take h so that $m \leq h \leq M$. (This is of importance in Fourier analysis, specially in the Riemann-Lebesgue lemma.)

12. **Egorov's theorem:** Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$, and assume that $f_k \to f$ a.e. on E. Given $\epsilon > 0$, we can find a closed set $A_\epsilon \subset E$ such that $M(E \backslash E_\epsilon) \leq \epsilon$ and $f_k \to f$ uniformly on A_ϵ.

Proof: Assume wlg (otherwise subtract a measure zero set) $f_k \to f, \forall x \in E$. For each pair of nonnegative integers n and k, let
$$E_k^n = \{x \in E : |f_j(x) - f(x)| < 1/n, \text{ for all } j > k\}.$$
Now fix n and note that $E_k^n \subset E_{k+1}^n$, and $E_k^n \uparrow E$ as $k \to \infty$. Since E has finite measure, exists k_n such that $m(E \backslash E_{k_n}^n) < 1/2^n$. By construction, we then have
$$|f_j(x) - f(x)| < 1/n \text{ whenever } j > k_n \text{ and } x \in E_k^n.$$
Choose N such that $\sum_{n=N}^{\infty} 2^{-n} < \epsilon/2$, and let
$$\widetilde{A}_\epsilon = \bigcap_{n \geq N} E_{k_n}^n.$$
First observe that
$$m(E \backslash \widetilde{A}_\epsilon) \leq \sum_{n=N}^{\infty} m(E \backslash E_{k_n}^n) < \epsilon/2.$$
Next, if $\delta > 0$, choose $n \geq N$ such that $1/n < \delta$ and note that $x \in \widetilde{A}_\epsilon$ implies $x \in E_{k_n}^n$. Therefore $|f_j(x) - f(x)| < \delta$ whenever $j > k_n$. Hence f_k converges uniformly to f on $\widetilde{A}_\epsilon$. Finally can choose a closed subset $A_\epsilon \subset \widetilde{A}_\epsilon$ with $m(\widetilde{A}_\epsilon \backslash A_\epsilon) < \epsilon/2$. Hence, $m(E \backslash A_\epsilon) < \epsilon$. □

13. **The Lebesgue integral of a nonnegative simple function:** A simple function f can be uniquely represented as $\sum_1^n c_i 1_{E_i}$ where $c_1, \ldots, c_n$ are the n distinct possible values of f, $E_i = f^{-1}(c_i)$ making E_i disjoint with $\cup E_i = \mathbb{R}$. For a **nonnegative** simple function uniquely represented as above the Lebesgue integral is defined as
$$\int f dm = \sum_1^n c_i m(E_i),$$
where the convention is that $0 \cdot \infty = 0$, i.e. if the value zero is assumed on a set of infinite measure then that contributes zero to the RHS.

It is well defined: If we also have $f = \sum_1^m d_j 1_{F_j}$, F_j's disjoint, $\cup F_j = \mathbb{R}$, then since f also assumes values $d_1, \ldots, d_m$, so each d_j is some c_i (say if $E_i \cap F_j$ nonempty, also implying $F_j \subset E_i$ since $E_i = f^{-1}(c_i)$). Hence $n \leq m$, some d_j's repeated, $E_i = \cup_{j:d_j=c_i} F_j$, giving $\sum_j d_j m(F_j) = \sum_i c_i \sum_{j:d_j=c_i} m(F_j) = \sum_i c_i m(E_i)$ since $m(E_i) = \sum_{j:d_j=c_i} m(F_j)$, F_j's being disjoint.

A function of the form $f = \sum_1^m d_j 1_{A_j}$, A_j's not necessarily disjoint, can be put in the above form using problems 1 and 2. Disjointify as follows: say for $m = 2$, $d_1(1_{A_1} + 0.1_{A_1^c}) \cdot (1_{A_2} + 1_{A_2^c}) + d_2(1_{A_1} + 1_{A_1^c}) \cdot (1_{A_2} + 0.1_{A_2^c})$, where in the jth term we take $1_{A_j} = (1_{A_j} + 0.1_{A_j^c}) \prod_{k \neq j}(1_{A_k} + 1_{A_k^c})$ to decompose 1_{A_j} as sum of indicators of the smallest (disjoint) sets of the algebra, summands sometimes multiplied by zero. Now collect sums of d_j's for the indicators of the disjoint parts.

M. Math. Measure Theory
Problems and Complements 3

1. (Good sets principle) Suppose Ω is a set, $\mathcal{C}$ is a class of subsets of Ω and $A \subset \Omega$. We denote by $\mathcal{C} \cap A$ the class $\{B \cap A : B \in \mathcal{C}\}$. If the minimal σ-algebra over $\mathcal{C}$ is $\sigma(\mathcal{C}) = \mathcal{F}$, then show that $\sigma_A(\mathcal{C} \cap A) = \mathcal{F} \cap A$, where $\sigma_A(\mathcal{C} \cap A)$ is the minimal σ-algebra of subsets of A over $\mathcal{C} \cap A$.

   Proof: $\mathcal{C} \subset \mathcal{F}$, hence $\mathcal{C} \cap A \subset \mathcal{F} \cap A$. Prove that $\mathcal{F} \cap A$ is a σ-algebra of subsets of A, thus $\sigma_A(\mathcal{C} \cap A) \subset \mathcal{F} \cap A$.

   For the other side we have to show $B \cap A \in \sigma_A(\mathcal{C} \cap A)$ for all $B \in \mathcal{F}$. Let $\mathcal{P}$ be the class of *good sets*, that is $\mathcal{P}$ consists of those sets $B \in \mathcal{F}$ such that $B \cap A \in \sigma_A(\mathcal{C} \cap A)$. Since $\mathcal{F}$ and $\sigma_A(\mathcal{C} \cap A)$ are σ-algebras, can show that $\mathcal{P}$ is a σ-algebra. But $\mathcal{C} \subset \mathcal{P}$, so $\sigma(\mathcal{C}) \subset \mathcal{P}$, hence $\mathcal{F} \subset \mathcal{P}$. Thus $\mathcal{F} = \mathcal{P}$ and the result follows. Briefly, every set in $\mathcal{C}$ is good and the class of good sets forms a σ-algebra. Consequently, every set in $\sigma(\mathcal{C})$ is good.

2. Let $f : \Omega \to \Omega'$ and let $\mathcal{C}$ be a class of subsets of Ω'. Show that $\sigma(f^{-1}(\mathcal{C})) = f^{-1}(\sigma(\mathcal{C}))$, where $f^{-1}(\mathcal{C}) = \{f^{-1}(A) : A \in \mathcal{C}\}$. (Use good sets principle.)

3. Now we restrict to functions defined on $\mathbb{R}$. Let f be measurable and B a Borel set. Then show that $f^{-1}(B)$ is a measurable set. (Hint: The class of sets for which $f^{-1}(E)$ is measurable is a σ-algebra containing the intervals.)

4. Show that if f is a measurable real valued function and g is a continuous function defined on $(-\infty, \infty)$, them $g \circ f$ is measurable.

   A **measure** on a σ-algebra $\mathcal{F}$ is a nonnegative extended real valued function μ on $\mathcal{F}$ such that whenever $A_1, A_2, \cdots$ form a finite or countably infinite collection of disjoint sets in $\mathcal{F}$, we have $\mu(\cup_n A_n) = \sum \mu(A_n)$.

5. Let Ω be a countably infinite set and let $\mathcal{F}$ consist of all subsets of Ω. Define $\mu(A) = 0$ if A is finite, $\mu(A) = \infty$ if A is infinite. (a) Show that μ is finitely additive but not countably additive. (b) Show that Ω is the limit of an increasing sequence of sets A_n with $\mu(A_n) = 0$ for all n, but $\mu(\Omega) = \infty$.

6. Let μ be the counting measure on Ω where Ω is an infinite set. Show that there is a sequence of sets $A_n \downarrow \phi$ with $\lim \mu(A_n) \neq 0$.

7. Let Ω be a countably infinite set and let $\mathcal{F}$ be the algebra consisting of all finite subsets of Ω and their complements. If A is finite set $\mu(A) = 0$ and if A^c is finite set $\mu(A) = 1$. (a) Show that μ is finitely additive but not countably additive on $\mathcal{F}$. (b) Show that Ω is the limit of an increasing sequence of sets $A_n \in \mathcal{F}$ with $\mu(A_n) = 0$ for all n, but $\mu(\Omega) = 1$.

   Integration

8. In the proof of Fatou's lemma, page 265 of Royden, one considers the function $f_k - (1-\epsilon)\phi$ where f_k is a (possibly extended real valued) nonnegative measurable function and ϕ is a nonnegative simple function. Why is this function measurable? (Such statements have been proved for real valued measurable functions only.)

9. Let f be a nonnegative measurable function. Show that $\int f = 0$ implies $f = 0$ a.e.

10. Let f be a nonnegative measurable function. (a) Show that there is an increasing sequence ϕ_n of nonnegative simple functions each of which vanishes outside a set of finite measure such that $f = \lim \phi_n$. (b) Show that $\int f = \sup \int \phi$ over all nonnegative simple functions $\phi \leq f$.

11. Let f be a nonnegative integrable function. Show that the function F defined by $F(x) = \int_{-\infty}^{x} f$ is continuous.

12. Let f_n be a sequence of nonnegative measurable functions that converge to f, and suppose $f_n \leq f$ for each n. Then show that $\int f = \lim \int f_n$.

13. (a) Show that we may have strict inequality in Fatou's lemma. Consider the sequence f_n defined by $f_n(x) = 1$ is $n \leq x < n+1$, with $f_n(x) = 0$ otherwise. (b) Show that the monotone convergence theorem need not hold for decreasing sequence of functions. Consider $f_n(x) = 0$ if $x < n$, $f_n(x) = 1$ if $x \geq n$.

14. Prove the following generalization of Fatou's lemma (often this is called Fatou's lemma): If f_n is a sequence of **nonnegative** functions, then

$$\int \liminf f_n \leq \liminf \int f_n$$

(Note that the other form follows from the above since if f_n converges to f a.e. then $\liminf f_n = \lim f_n$.

15. Let f_n be a sequence of nonnegative measurable functions on $(-\infty, \infty)$ such that $f_n \to f$ a.e., and suppose that $\int f_n \to \int f < \infty$. Then show that for each measurable set E we have $\int_E f_n \to \int_E f$. (Fatou gives $\int_E f \leq \liminf \int_E f_n$, $\int_{E^c} f \leq \liminf \int_{E^c} f_n$. Manipulate the second inequality.)

16. Consider $f_n = 1_{\{r_1,\ldots,r_n\}}$ where $r_1, r_2, \ldots$ is an enumeration of rationals on $Q \cap [0,1]$. This f_n converges everywhere to $1_{Q\cap[0,1]}$. (a) Show that $g_n = 1 - f_n$ defined on $[0,1]$ is Riemann integrable but $\lim g_n = 1_{Q^c\cap[0,1]} = g$, say, is not Riemann integrable. (b) If you use Lebesgue integrals which theorem will enable you to interchange limit and integration and what will be $\int_{[0,1]} g\, dm$?

17. Proposition 4, Page 81, Royden 3rd ed: Let f be a bounded function on $[a,b]$. If f is Riemann integrable then it is measurable and $R\int_a^b f(x)dx = \int_{[a,b]} f\, dm$.

    Proof: Since every step function is also a simple function we have

$$R\underline{\int_a^b} f(x)dx \leq sup_{\phi\leq f} \int_{[a,b]} \phi\, dm \leq \inf_{\psi\geq f} \int_{[a,b]} \psi\, dm \leq R\overline{\int_a^b} f(x)dx.$$

As f is Riemann integrable the two extremes are equal. Since f is bounded, $C < f < D$ say, translating by C or D as needed we'll later prove that the middle terms are $\int_{[a,b]} f\, dm$.

First use part of Proposition 3, page 80-81 to get measurability: Since $\inf_{\psi\geq f} \int_{[a,b]} \psi dm = sup_{\phi\leq f} \int_{[a,b]} \phi dm$ there exist $\phi_n \leq f \leq \psi_n$ such that $\int_{[a,b]}(\psi_n - \phi_n)dm < \frac{1}{n}$. Denote $sup_n\phi_n = \phi^\star \leq f \leq \psi^\star = inf_n\psi_n$. Consider the set $\{\psi^\star - \phi^\star > \frac{1}{\nu}\} \subset \{\psi_n - \phi_n > \frac{1}{\nu}\}$. Since $\frac{1}{n} > \int_{[a,b]}(\psi_n - \phi_n)dm \geq \int_{[a,b]}(\psi_n - \phi_n)1_{\{\psi_n-\phi_n>1/\nu\}}\, dm \geq \int_{[a,b]} \frac{1}{\nu}1_{\{\psi_n-\phi_n>1/\nu\}}\, dm = \frac{1}{\nu}m(\{\psi_n - \phi_n > 1/\nu\})$ it follows that $m(\{\psi^\star - \phi^\star > \frac{1}{\nu}\}) \leq \nu/n$ and is actually zero since n is arbitrary. Since $\{\psi^\star - \phi^\star > 0\} = \cup_\nu\{\psi^\star - \phi^\star > \frac{1}{\nu}\}$, $\psi^\star - \phi^\star$ is zero almost surely and f equals the measurable function $\phi^\star$ or $\psi^\star$.

Now if f is bounded measurable then one also has $\int_{[a,b]} f dm = \sup_{\phi\leq f} \int \phi\, dm = \inf_{\psi\geq f} \int \psi\, dm$. To prove this, suppose $C < f < D$. Notice that $\sup_{\phi\leq f} \int \phi\, dm$ will occur over $\sup_{C\leq\phi\leq f}$. Now $0 < f - C$, so $\sup_{0\leq\phi-C\leq f-C} \int_{[a,b]}(\phi - C)dm = \int_{[a,b]}(f - C)dm$, which after cancelling $m([a,b])$ is $\sup_{C\leq\phi\leq f} \int_{[a,b]} \phi\, dm = \int_{[a,b]} f dm$. Similarly try the other one with $D - f > 0$, but now sup of a negative turning negative of inf.

M. Math. Measure Theory
Problems and Complements 4

1. Show that if f is integrable over E then so is $|f|$ and $|\int_E f| \le \int_E |f|$. Does the integrability of $|f|$ imply that of f?

2. If f is nonnegative and $\int f\, dm < \infty$ then $f < \infty$ a.e. To do this directly consider the set $\{f > n\}$ and using $f \ge n1_{\{f>n\}}$ show that $m(\{f > n\}) < \frac{\int f\, dm}{n} \to 0$. However $\{f > n\} \downarrow \{f = \infty\}$.

3. (Proposition 15, page 267 of Royden) If f and g are integrable functions and E is a measurable set then (i) $\int_E (c_1 f + c_2 g)\, dm = c_1 \int f\, dm + c_2 \int g\, dm$, (ii) if h is measurable and $|h| \le |f|$ then h is integrable, (iii) if $f \ge g$ a.e. then $\int f\, dm \ge \int g\, dm$.

4. (a) Let g be a nonnegative measurable function on $\mathbb{R}$. Set $\nu(E) = \int_E g dm$. Show that ν is a measure on $\mathcal{M}$. (b) Let f be a nonnegative measurable function on $\mathbb{R}$. Then $\int_E f d\nu = \int_E f g dm$. (First establish this for the case of f simple and then use the monotone convergence theorem taking $\phi_n \uparrow f$. This problem is a little beyond the development since we have not done integration wrt other measures yet, but such integration wrt ν replaces m by ν everywhere and reproves Fatou, MCT, DCT.)

5. The improper Riemann integral of a function may exist without the function being integrable (in the sense of Lebesgue), e.g. if $f(x) = \sin x/x$ on $[0, \infty]$. However if f is integrable show that the improper Riemann integral is equal to the Lebesgue integral whenever the former exists (recall, for bounded intervals the Riemann integral of a bounded function is equal to its Lebesgue integral).

6. (a) Let g be an integrable function on a set E and suppose that f_n is a sequence of measurable functions such that $|f_n(x)| \le g(x)$ a.e. on E. Then $\int_E \liminf f_n \le \liminf \int_E f_n \le \limsup \int_E f_n \le \int_E \limsup f_n$. (b) Suppose $f_n : \mathbb{R} \to [0, \infty]$ is measurable and $f_1 \ge f_2 \ge f_3 \ge \cdots, f_n(x) \to f(x) \forall x \in \mathbb{R}$ and $\int_{\mathbb{R}} f_1 < \infty$. Then show that $\lim \int f_n = \int f$. Compare with a previous problem about the invalidity of MCT for decreasing sequence of functions.

7. Let g_n be a sequence of integrable functions which converges a.e. to an integrable function g. Let f_n be a sequence of measurable functions such that $|f_n| \le g_n$ and f_n converges to f a.e. If $\int g = \lim \int g_n$ then $\int f = \lim \int f_n$.

8. Under the same set up show that $\int |f_n - f| \to 0$.

9. Let f_n be a sequence of integrable functions such that $f_n \to f$ a.e. with f integrable. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$. ($|\int (|f_n| - |f|)| \le \int |f_n - f|$.)

10. (a) Let f be integrable over E. Then given $\epsilon > 0$ there is a simple function ϕ such that $\int |f - \phi| < \epsilon$. (Deal with the positive and negative parts separately, for each approach through simple functions from below.) (b) Under the same hypothesis there is a step function ψ suxch that $\int_E |f - \psi| < \epsilon$. (Enough to approximate each indicator function by a step function and then apply triangle inequality.) (c) Under the same hypothesis, there is a continuous function g vanishing outside a finite interval such that $\int_E |f - g| < \epsilon$.

11. Establish the Riemann-Lebesgue lemma: If f is an integrable function on $(-\infty, \infty)$, then $\lim_{n\to\infty} f(x) \cos nx\, dx = 0$. (Hint: First prove for step functions, namely for the indicator of an interval, then use the previous problem.)

12. (a) Let f be integrable over $(-\infty, \infty)$. Then $\int f(x)dx = \int f(x + t)dx$. (start with indicators.) (b) Let g be a bounded measurable function. Then $\lim_{t\to 0} \int_{-\infty}^{\infty} |g(x)[f(x) - f(x + t)]| = 0$. (If f is continuous and vanishes outside a finite interval, then can use uniform continuity.)

13. Let f be a function of two variables (x, t) defined on $Q = [0, 1] \times [0, 1]$ and which is a measurable function of x for each fixed value of t. Suppose that $\lim_{t\to 0} f(x, t) = f(x)$ and that for all t we have $|f(x, t)| \le g(x)$, where g is an integrable function on $[0, 1]$. Then $\lim_{t\to 0} \int f(x, t)dx = \int f(x)dx$. Show also that if the function $f(x, t)$ is continuous in t for each x, then $h(t) = \int f(x, t)dx$ is a continuous function of t.

14. Let f be a function defined and bounded on $Q = [0,1] \times [0,1]$ and suppose that for each fixed t the function f is a measurable function of x. For each $(x,t) \in Q$, let the partial derivative $\partial f/\partial t$ exist and it is bounded in Q. Then

$$\frac{d}{dt}\int_0^1 f(x,t)dx = \int_0^1 \frac{\partial f}{\partial t}dx.$$

15. Suppose $f : \mathbb{R} \to [0,\infty]$ is measurable and $\int_{\mathbb{R}} f = c, 0 < c < \infty$ and α is a constant. Show that

$$\lim_{n\to\infty}\int_{\mathbb{R}} n\log[1 + (f/n)^\alpha]$$

equals (a) ∞, if $0 < \alpha < 1$, (b) c, if $\alpha = 1$, (c) 0, if $1 < \alpha < \infty$. (Hint: If $\alpha \geq 1$, the integrands are dominated by αf. If $\alpha < 1$, Fatou's lemma can be applied.)

16. Show that (a) $\int_1^\infty e^{-t}\log t = \lim \int_1^n [1-(t/n)]^n \log t\,dt$, (b) $\int_0^1 e^{-t}\log t\,dt = \lim \int_0^1 [1-(t/n)]^n \log t\,dt$. ($e^{-t}$ can be used for domination.)

17. One can prove MCT directly first and then deduce Fatou from it. For this approach see Rudin, Real and Complex Analysis, page 21-22 and page 23. The proof uses similar ideas.

M. Math. Measure Theory
Problems and Complements 5

L^p spaces

1. Let f be a bounded measurable function on $[0,1]$. Then $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

2. If $f \in L^1$ and $g \in L^\infty$ then $\int |fg| \leq ||f||_1 \cdot ||g||_\infty$.

3. (a) For $1 \leq p < \infty$, we denote by l^p the space of all sequences $\xi_v, \nu = 1, 2, \cdots$ such that $\sum_{\nu=1}^\infty |\xi_\nu|^p < \infty$. Prove the Minkowski inequality for sequences:

$$||\{\xi_\nu + \eta_\nu\}||_p \leq ||\{\xi_\nu\}||_p + ||\{\eta_\nu\}||_p,$$

where the p-th norm is given by $(||\{\xi_\nu\}||_p)^p = \sum_{\nu=1}^\infty |\xi_\nu|^p$ and for $p = \infty$ we take $\sup |\xi_\nu|$.

(b) Establish Holder's inequality for sequences: if $\{\xi_\nu\} \in l^p, \{\eta_\nu\} \in l^q$ with p, q conjugate, then $\sum_{\nu=1}^\infty |\xi_\nu \eta_\nu| \leq ||\{\xi_\nu\}||_p \cdot ||\{\eta_\nu\}||_q$.

When we do measures on more general spaces, the above will appear as a particular case, namely counting measure on the integers and the L^p spaces for that measure.

4. Let f_n be a sequence of functions in L^∞. Prove that f_n converges to f in L^∞ iff there is a set E of measure zero such that f_n converges to f uniformly on E^c.

5. Prove that L^∞ is complete (at each point the Cauchy criterion for real numbers applies).

6. Prove that l^p is complete $(1 \leq p < \infty)$.

7. Let $C = C[0,1]$ be the space of all continuous functions on $[0,1]$ and define $||f|| = \max |f(x)|$. Show that C is a Banach space (you only have to recall the limit of a uniformly convergent seuqence of continuous functions).

8. Denote by l^∞ the space of all bounded sequences of real numbers and define $||\{\xi_\nu\}||_\infty = \sup |\xi_\nu|$. Show that l^∞ is a Banach space.

9. Show that the space c of all convergent sqeuences of real numbers and the space c_0 of all sequences which converge to zero are Banach spaces (with the l^∞ norm).

10. Let f_n be a sequence of functions in $L^p, 1 \leq p < \infty$, which converge a.e. to a function $f \in L^p$. Show that f_n converges to f in L^p iff $||f_n||_p \to ||f||_p$. (We did this as a problem in DCT for $p = 1$. Here use Fatou's lemma for one side and Minkowski's inequality to get limsup.)

11. Let f_n be a sequence of functions in $L^p, 1 < p < \infty$, which converge a.e. to a function $f \in L^p$. and suppose that there is a constant M such that $||f_n|| \leq M$ for all n. Then for each function $g \in L^q$ we have $\int fg = \lim \int f_n g$. (On f_n use Minkowski's inequality as in the previous problem to get limsup. This problem may need Egorov's theorem which we haven't done yet.))

12. Let $f_n \to f$ in $L^p, 1 \leq p < \infty$, and let g_n be a sequence of measurable functions such that $|g_n| \leq M$ for all n, and $g_n \to g$ a.e. Then $g_n f_n \to gf$ in L^p. $(g_n f_n - gf = g_n(f_n - f) + (g_n - g)f$. You know many inequalities and convergence theorems.)

13. Here we work out the completeness of l^p spaces, $1 \leq p \leq \infty$:

First for $p = \infty$, consider $\boldsymbol{f}_m = (f_m(1), f_m(2), \cdots)$ under $||\boldsymbol{f}_m||_\infty = \sup_i |f_m(i)|$. If $\{\boldsymbol{f}_m\}$ is Cauchy (under the above norm), then for each i, $|f_n(i) - f_m(i)| \leq ||\boldsymbol{f}_n - \boldsymbol{f}_m||_\infty < \epsilon$ for $n \geq m \geq N_\epsilon$. So $\{f_n(i)\}$ is a Cauchy sequence of reals, hence converges to some $f(i)$ say. Denote $\boldsymbol{f} = (f(1), f(2), \ldots)$.

1. TST $\boldsymbol{f} \in l^\infty$. Since for each i, $|f(i)| = \lim_n |f_n(i)| \leq \limsup_n |f_n(i) - f_m(i)| + |f_m(i)| \leq \limsup_n ||\boldsymbol{f}_n - \boldsymbol{f}_m||_\infty + ||\boldsymbol{f}_m||_\infty \leq \epsilon + ||\boldsymbol{f}_m||_\infty$, where $m \geq N_\epsilon$ is fixed, we have $||\boldsymbol{f}||_\infty < \infty$.

2. TST $\boldsymbol{f}_m \to \boldsymbol{f}$ in $||.||_\infty$ norm. For each i, $|f_m(i)-f(i)| = \lim_n |f_m(i)-f_n(i)| \leq \limsup_n ||\boldsymbol{f}_m - \boldsymbol{f}_n||_\infty < \epsilon$ if $m > N_\epsilon$, fixed. Thus $m \geq N_\epsilon$ imples $||\boldsymbol{f}_m - \boldsymbol{f}||_\infty < \epsilon$. $\square$

Next for $1 \leq p < \infty$, under $||\boldsymbol{f}_m||_p = \{\sum_1^\infty |f_m(i)|^p\}^{1/p}$, since $|f_n(i) - f_m(i)| \leq ||\boldsymbol{f}_n - \boldsymbol{f}_m||_p$ (check), get each $f_n(i) \to f(i)$, giving $\boldsymbol{f}$ as before.

1. TST $\boldsymbol{f} \in l^p$. Note that for each fixed k, $\{\sum_{i=1}^k |f_n(i)|^p\}^{1/p} \leq \{\sum_1^\infty |f_n(i)|^p\}^{1/p} = ||\boldsymbol{f}_n||_p \leq ||\boldsymbol{f}_n - \boldsymbol{f}_m||_p + ||\boldsymbol{f}_m||_p$. Fix $m \geq N_\epsilon$, make $n \uparrow \infty$. LHS has a limit, a limsup on the right gives $\{\sum_{i=1}^k |f(i)|^p\}^{1/p} \leq \epsilon + ||\boldsymbol{f}_m||_p$. Making $k \uparrow \infty$ on the LHS get $\sum_1^\infty |f(i)|^p < \infty$, i.e. $\boldsymbol{f} \in l^p$.

2. As an exercise try $||\boldsymbol{f}_m - \boldsymbol{f}||_p < \epsilon$, if $m \geq N_\epsilon$. First consider the sum upto k ...

For $L^p(\mathbb{R}), 1 \leq p < \infty$, the Riesz Fischer theorem has been proved. But for $L^\infty(\mathbb{R})$ one needs some a.e. considerations. For each x, $|f_n(x) - f_m(x)| \leq ||f_n - f_m||_\infty$ outside $E_{n,m}$ where $m(E_{n,m}) = 0$. To make all these inequalities over all n, m work consider $E = \cup_{n,m} E_{n,m}$, which also has measure zero. Now over E^c repeat the proof of l^∞ but with x instead of i to get $|f(x)| \leq \epsilon + ||f_m||_\infty$. Take sup over $x \in E^c$, to get $||f||_\infty < \infty$. Try the other part over E^c similarly.

M. Math. Measure Theory
Problems and complements 6

1. Let $(X, \mathcal{B}, \mu)$ be a measure space and $Y \in \mathcal{B}$. Let $\mathcal{B}_Y$ consist of those sets of $\mathcal{B}$ that are contained in Y. Set $\mu_Y(E) = \mu(E)$ if $E \in \mathcal{B}_Y$. Then $(Y, \mathcal{B}_Y, \mu_Y)$ is a measure space and μ_Y is called the restriction of μ to Y.

2. If $(X, \mathcal{B}, \mu)$ is a measure space, then we can find a complete measure space $(X, \mathcal{B}_0, \mu_0)$ such that (a) $\mathcal{B} \subset \mathcal{B}_0$, (b) $E \in \mathcal{B} \Rightarrow \mu(E) = \mu_0(E)$, (c) $E \in \mathcal{B}_0 \Leftrightarrow E = A \cup B$ where $B \in \mathcal{B}$ and $A \subset C, C \in \mathcal{B}, \mu(C) = 0$.

   Hint: 1. First show tht $\mathcal{B}_0$ defined by (c) is a σ-algebra. The only problem is complementation, $A^c \cap B^c$. Note that $B^c = C^c \cup (C \backslash B)$, so that $A^c \cap B^c = (A^c \cap C^c) \cup (A^c \cap (C \backslash B))$, but the second one is a subset of C. 2. Now, if $E \in \mathcal{B}_0$, show that $\mu(A)$ is the same for all sets $A \in \mathcal{B}$ such that $E = A \cup B$ with B a subset of a set of measure zero (suppose $A_1 \cup B_1 = A_2 \cup B_2$ with sets as defined. Then from $A_1 \cup C_1 = A_1 \cup B_1 \cup (C_1 \backslash B_1) = A_2 \cup B_2 \cup (C_1 \backslash B_1)$ show $\mu(A_1) = \mu(A_1 \cup C_1) \leq \mu(A_2) + \mu(C_2) + \mu(C_1) = \mu(A_2)$ and conversely). Use this fact to define $\mu_0(E) = \mu(A)$ and show μ_0 is a measure.

3. Given an outer measure $\mu^\star$ on all subsets of X consider the class $\mathcal{B}$ of $\mu^\star$ measurable sets. If $\bar{\mu}$ is $\mu^\star$ restricted to $\mathcal{B}$, then show that $\bar{\mu}$ is complete. (Suppose $E \subset B, B \in \mathcal{B}$ with $\mu^\star(B) = \bar{\mu}(B) = 0$. Then for any A, $\mu^\star(A \cap E) + \mu^\star(A \cap E^c) \leq \mu^\star(B) + \mu^\star(A) = \mu^\star(A)$. The other side follows from properties of outer measure. So E is measurable.) The point is that later we extend a measure on an algebra $\mathcal{A}$ to the smallest σ-algebra containing $\mathcal{A}$, and not to the class of all measurable sets. In that case it may not be the case that the extension is complete on the smallest σ-algebra containing $\mathcal{A}$. However, one can get another measure space that will be complete, from the previous problem. You should also compare Lebesgue measure on Borel sets and measurable sets.

4. Assume that E_i is a sequence of disjoint measurable sets and $E = \cup E_i$. Then for any set A we have $\mu^\star(A \cap E) = \sum \mu^\star(A \cap E_i)$.

5. A collection $\mathcal{C}$ of subsets of X is called a semialgebra if (a) the intersection of any two sets in $\mathcal{C}$ is in $\mathcal{C}$ and (b) the complement of any set in $\mathcal{C}$ is a finite disjoint union of sets in $\mathcal{C}$ (think of intervals of the form $(a, b]$ and their intersection, complement etc).

   (a) Let $\mathcal{C}$ be a semialgebra of sets and $\mathcal{A}$ be the smalllest algebra of sets containing $\mathcal{C}$. Show that $\mathcal{A}$ consists of sets of the form $A = \cup_{i=1}^n C_i$ with $C_i \in \mathcal{C}$. Show that $\mathcal{A}_\sigma = \mathcal{C}_\sigma$.

   (b) Let $\mathcal{C}$ be a semialgebra of sets containing ϕ and μ a nonnegative set function defined on $\mathcal{C}$ with $\mu(\phi) = 0$. Then μ has a unique extension to a measure on the algebra $\mathcal{A}$ generated by $\mathcal{C}$ if the following conditions are satisfied (1) If a set $C \in \mathcal{C}$ is the union of a finite disjoint collection $\{C_i\}$ of sets in $\mathcal{C}$, then $\mu(C) = \sum \mu(C_i)$. (2) If a set $C \in \mathcal{C}$ is the union of a countable disjoint collection $\{C_i\}$ of sets in $\mathcal{C}$ then $\mu(C) \leq \sum \mu(C_i)$.

   Hint: (1) implies that if A is the union of each of two finite disjoint collections $\{C_i\}$ and $\{D_j\}$ of sets in $\mathcal{C}$ then $\sum \mu(C_i) = \sum \mu(D_j)$ since $\mu(C_i) = \sum_j \mu(C_i \cap D_j)$ and the collections are finite. Now, condition (2) implies μ is countably additive on $\mathcal{A}$, since finite additivity and monotonicity already imply the reverse inequality.

6. Let μ be a finite measure on an algebra $\mathcal{A}$ and $\mu^\star$ the induced outer measure. Show that a set E is measurable iff for each $\epsilon > 0$ there is a set $A \in \mathcal{A}_\sigma, A \subset E$, such that $\mu^\star(E \backslash A) < \epsilon$.

7. An outer measure $\mu^\star$ is said to be regular if given any subset E of X and any $\epsilon > 0$, there is a $\mu^\star$-measurable set A with $E \subset A$ and $\mu^\star(A) \leq \mu^\star(E) + \epsilon$. Show that every outer measure induced by a measure on an algebra $\mathcal{A}$ is a regular outer measure. (1. Sets in $\mathcal{A}$ are measurable. 2. Approximate by by $\mathcal{A}_\sigma$ sets from above.)

8. Let μ be a measure on an algebra $\mathcal{A}$ and $\bar{\mu}$ the extension of it given by the Caratheodory process. Let E be measurable wrt $\bar{\mu}$ and $\bar{\mu}(E) < \infty$. Then given $\epsilon > 0$, there is an $A \in \mathcal{A}$ with $\bar{\mu}(A \Delta E) < \epsilon$. (We did it for Lebesgue measure in Problem Set 1, repeat the proof in this case. This construction is then used in approximating measurable functions by functions constant on rectangles in the L^p sense.)

9. Let F be the culultative distribution function of the measure ν and assume that F is continuous. Then for any Borel set contained in the range of F we have $m(E) = \nu(F^{-1}(E))$ where m denotes Lebesgue measure. (For any interval in the range of F, $\nu(F^{-1}(c,d)) = \nu(F^{-1}(c), F^{-1}(d)) = F(F^{-1}(d)) - F(F^{-1}(c)) = d - c$, now extend by Caratheodory extension theorem.) Generalize to discontinuous cumulative distribution functions.

10. Let F be a monotone increasing function and define $F^\star(x) = \lim_{y \to x^+} F(y)$. Then $F^\star$ is a monotone increasing function which is continuous on the right and agrees with F whenever F is continuous on the right. We have $(F^\star)^\star = F^\star$ and if F and G are monotone increasing functions which agree whenever they are both continuous then $F^\star = G^\star$.

11. Let f be a nonnegative measurable function. Then there is a sequence ϕ_n of simple functions with $\phi_{n+1} \geq \phi_n$ such that $\lim \phi_n = f$ at each point of X. If f is defined on a σ-finite measure space then we may choose the functions ϕ_n so that each vanishes outside a set of finite measure. (For each pair (n, k) of integers define
$$E_{n,k} = \{x : k2^{-n} \leq f(x) < (k+1)2^{-n}\},$$
and set $\phi_n = 2^{-n} \sum_{k=0}^{2^{2n}} k 1_{E_{n,k}}$.

12. If μ is a complete measure and f is a measurable function, then $f = g$ a.e. implies g is measurable. (We did it for Lebesgue measure.)

13. Let f_n be a sequence of measurable functions that converge to a function except at the points of a set E of measure zero. Then f is a measurable function if μ is complete.

14. A sequence of measurable real valued functions f_n is said to converge in measure to f if given $\epsilon > 0$, there is an integer N and a measurable set E with $\mu(E) < \epsilon$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and all $x \notin E$.

    (a) Show that if f_n converges to f in measure then there is a subsequence f_{n_k} that converges to f almost everywhere. (Given ν, there is an integer n_ν such that for all $n \geq n_\nu$ we have $\mu(E_\nu) = \mu(x : |f_n(x) - f(x)| \geq 2^{-\nu}) < 2^{-\nu}$. If $x \notin \cup_{\nu=k}^{\infty} E_\nu$, we must have $|f_{n_\nu}(x) - f(x)| < 2^{-\nu}$ for $\nu \geq k$, and so $f_{n_\nu}(x) \to f(x)$. and also for any $x \notin A = \cap_{k=1}^{\infty} \cup_{\nu=k}^{\infty} E_\nu$. But $\mu(A) \leq \mu(\cup_{\nu=k}^{\infty} E_\nu) \leq \sum_{\nu=k}^{\infty} \mu(E_\nu) = 2^{-k+1}$ for any k. Thus $\mu(A) = 0$.)

    (b) Suppose f_n is a sequence of measurable functions each of which vanishes outside a fixed measurable set A with $\mu(A) < \infty$. Suppose that $f_n(x) \to f(x)$ for x a.e. Then f_n converges to f in measure.

    (c) A sequence of measurable functions f_n is said to be Cauchy in measure if given $\epsilon > 0$, there is an integer N and a measurable set E with $\mu(E) < \infty$ such that $|f_n(x) - f_m(x)| < \epsilon$ for all $m, n \geq N$ and all $x \notin E$. Show that if f_n is Cauchy in measure then there is a function f to which f_n converges in measure. (Choose $n_{\nu+1} > n_\nu$ so that $\mu(x : |f_{n_\nu} - f_{n_{\nu+1}}| < 2^{-\nu}) < 2^{-\nu}$. Then the series $\sum (f_{n_{\nu+1}} - f_{n_\nu})$ converges a.e. to a function g. Let $f = g + f_{n_1}$. Then $f_{n_\nu} \to f$ in measure and consequently one can show $f_n \to f$ in measure.

M. Math. Measure Theory
Problems and complements 7

1. Suppose that μ is not complete but that we define a bounded function f to be integrable over a set E of finite measure if
$$\sup_{\phi \le f} \int_E \phi d\mu = \inf_{\psi \ge f} \int_E \psi d\mu$$
for all simple functions ϕ and ψ. Show that f is integrable iff it is measurable wrt the completion of μ. (We did it for Lebesgue measure, see page 79-81 of Royden, third ed.)

2. Let f be an integrable function on the measure space $(X, \mathcal{B}, \mu)$. Show that given $\epsilon > 0$, there is a $\delta > 0$, such that for each measurable set E with $\mu(E) < \delta$ we have $|\int_E f d\mu| < \epsilon$.

3. Consider $\mathbb{R}^n$ with n-dimensional Lebesgue measure. If $f \in L^1(\mathbb{R}^n)$ then there exists bounded continuous g such that $\int |f - g| < \epsilon$. (Enough to do this for 1_E of finite measure. Get disjoint (closed) rectangles $I_j, j = 1, 2, \cdots, k$ such that $E\Delta(\cup_{j=1}^k I_j)$ has small measure. Using an open cover get a finite cover for $\cup_{j=1}^k I_j$, take complement and use Urysohn's lemma to get a continuous function.)

4. (a) Show that if f is integrable, then the set $\{x : f(x) \ne 0\}$ is of σ-finite measure. (b) Show that if f is integrable, $f \ge 0$, then $f = \lim \phi_n$ for some increasing sequence of simple functions each of which vanishes outside a set of finite measure. (c) Show that if f is integrable wrt μ then given $\epsilon > 0$ there is a simple function ϕ such that $\int |f - \phi| d\mu < \epsilon$.

5. (a) Let $(X, \mathcal{B}, \mu)$ be a measure space and g a nonnegative measurable function on X. Set $\nu(E) = \int_E g d\mu$. Show that ν is a measure on $\mathcal{B}$. (b) Let f be a nonnegative measurable function on X. Then $\int f d\nu = \int f g d\mu$. (First establish this for the case of f simple and then use the monotone convergence theorem.)

6. Let $X = Y = [0, 1]$, and let $\mu = \nu$ be the Lebesgue measure. Show that each open set in $X \times Y$ is measurable, and hence each Borel set in $X \times Y$ is measurable.

7. Let h and g be integrable functions on X and Y and define $f(x, y) = h(x)g(y)$. Then f is integrable on $X \times Y$ and $\int_{X \times Y} f d(\mu \times \nu) = \int_X h d\mu \int_Y g d\nu$. (Here σ-finiteness of the measures has not been assumed, thus you cannot apply Tonelli. Try simple functions and convergence theorems.)

8. Show that Tonelli's theorem is still true if instead of assuming μ and ν to be σ-finite we merely assume that $\{(x, y) : f(x, y) \ne 0\}$ is a set of σ-finite measure.

9. The following example shows that one cannot remove the assumption that f be nonnegative from the Tonelli theorem or that f be integrable from the Fubini theorem. Let $X = Y$ be the positive integers and $\mu = \nu$ be the counting measure. Let
$$\begin{aligned} f(x, y) &= 2 - 2^{-x} \text{ if } x = y \\ &= -2 + 2^{-x} \text{ if } x = y + 1 \\ &= 0 \text{ otherwise.} \end{aligned}$$

10. The following example shows that we cannot remove the hypothesis that f be integrable or that μ and ν are σ-finite from the Tonelli theorem. Let $X = Y = [0, 1]$ with $\mathcal{A} = \mathcal{B}$ the class of Borel sets. Let μ be Lebesgue measure and ν the counting measure. Then the diagonal $\Delta = \{(x, y) : x = y\}$ is measurable (is an $\mathcal{R}_{\sigma\delta}$ in fact), but its indicator function fails to satisfy any of the equalities between iterated and product integrals.

11. The smallest σ-algebra containing measurable rectangles coming from two σ-algebras $\mathcal{A}, \mathcal{B}$ is denoted by $\mathcal{A} \times \mathcal{B}$. Note that we have not talked about measure here and were there two measures μ, ν this σ-algebra could be smaller than the σ-algebra on which the completion $\mu \times \nu$ is defined. (a) Show that if $E \in \mathcal{A} \times \mathcal{B}$ then $E_x \in \mathcal{B}$ for each x. (Fix x, and show that such E's form a σ-algebra containing the measurable rectangles.) (b) If f is measurable wrt $\mathcal{A} \times \mathcal{B}$ then f_x is measurable wrt $\mathcal{B}$ for each x.

12. Let $X = Y = \mathbb{R}$ and $\mu = \nu =$ Lebesgue measure, so that $\mu \times \nu$ is the two dimensional Lebesgue measure on $\mathbb{R}^2$ denoted by $dxdy$. (a) For each measurable subset E of $\mathbb{R}$ the set $\{(x, y) : x - y \in E\}$ is a measurable subset of $\mathbb{R}^2$ (what kind of function is $(x, y) \mapsto x - y$? Try E Borel, E of measure zero, etc.) (b) If f is a measurable function on $\mathbb{R}$ then $F(x, y) = f(x - y)$ is a measurable function on $\mathbb{R}^2$. (c) If f and g are integrable functions on $\mathbb{R}$, then for almost all x the function ϕ defined by $\phi(y) = f(x-y)g(y)$ is integrable. If we denote its integral by $h(x)$ then h is integrable and $\int |h| \leq \int |f| \int |g|$.

13. Let f and g be functions in $L^1(\mathbb{R})$ and define $f \star g$ to be the function $h(y) = \int f(y - x)g(x)dx$. (a) Show that $f \star g = g \star f$. (b) Show that $(f \star g) \star \phi = f \star (g \star \phi)$ where $\phi \in L^1(\mathbb{R})$. (c) For $f \in L^1(\mathbb{R})$ define $\hat{f}$ by $\hat{f}(s) = \int e^{ist} f(t)dt$. Then $\hat{f}$ is a boundeed complex function and $\widehat{f \star g} = \hat{f}\hat{g}$.

14. Let f be a nonnegative integrable function on $\mathbb{R}$ and let m_2 be two dimensional Lebesgue measure on $\mathbb{R}^2$. (a) Then $m_2\{(x, y) : 0 \leq y \leq f(x)\} = m_2\{(x, y) : 0 < y < f(x)\} = \int f(x)$. (b) Let $\phi(t) = m\{x : f(x) \geq t\}$. Then ϕ is a decreasing function and $\int_0^\infty \phi(t)dt = \int f(x)dx$.

15. If $(X_i, \mathcal{A}_i, \mu_i)_{i=1}^n$ is a finite collection of measure spaces we can form the product measure $\mu_1 \times \cdots \times \mu_n$ on the space $X_1 \times \cdots \times X_n$ by starting with the semialgebra of rectangles of the form $R = A_1 \times \cdots \times A_n$ and $\mu(R) = \Pi \mu_i(A_i)$. and using the Caratheodory extension procedure. Show that if we identify $(X_1 \times \cdots \times X_p) \times (X_{p+1} \times \cdots \times X_n) = X_1 \times \cdots \times X_n$ then $(\mu_1 \times \cdots \times \mu_p) \times (\mu_{p+1} \times \cdots \times \mu_n) = \mu_1 \times \cdots \times \mu_n$.

16. Use Fubini's theorem and the relation $1/x = \int_0^\infty e^{-xt}dt, x > 0$, to prove that $\lim_{A\to\infty} \int_0^A (\sin x/x)dx = \pi/2$.

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Problems and complements 8

1. Show that there is only one pair of mutually singular measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$.

2. Show that if E is any measurable set then $-\nu^-(E) \leq \nu(E) \leq \nu^+(E)$ and $|\nu(E)| \leq |\nu|(E)$.

3. Show that if ν_1 and ν_2 are any two finite signed measures, then so is $\alpha\nu_1 + \beta\nu_2$ where α, β are real numbers. Show that $|\alpha\nu| = |\alpha||\nu|$ and $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$, where $\nu \leq \mu$ means $\nu(E) \leq \mu(E)$ for all measurable set E.

4. We define integration wrt a signed measure by defining $\int f d\nu = \int f d\nu^+ - \int f d\nu^-$. If $|f| \leq M$ then $|\int_E f d\nu| \leq M|\nu|(E)$. Moreover, there is a measurable function f with $|f| \leq 1$ such that $\int_E f d\nu = |\nu|(E)$.

5. Show that the Radon-Nikodym theorem for a finite measure μ implies the theorem for a σ-finite measure μ. Show the uniqueness of the function f in the Radon-Nikodym theorem.

6. (a) Let $(X, \mathcal{B}, \mu)$ be a measure space and g a nonnegative measurable function on X. Set $\nu(E) = \int_E g d\mu$. Show that ν is a measure on $\mathcal{B}$. (b) Let f be a nonnegative measurable function on X. Then $\int f d\nu = \int f g d\mu$. (First establish this for the case of f simple and then use the monotone convergence theorem.)

7. Let μ, ν and λ be σ-finite. Show that the Radon-Nikodym derivative $d\nu/d\mu$ has the following properties: (a) If $\nu \ll \mu$ and f is a nonnegative measurable function then $\int f d\nu = \int f (d\nu/d\mu) d\mu$, (b) $d(\nu_1 + \nu_2)/d\mu) = d\nu_1/d\mu + d\nu_2/d\mu$, (c) if $\nu \ll \mu \ll \lambda$, then $d\nu/d\lambda = (d\nu/d\mu)(d\mu/d\lambda)$, (d) if $\nu \ll \mu$ and $\mu \ll \nu$, (this is also referred to as μ and ν are equivalemt measures) then $d\nu/d\mu = (d\mu/d\nu)^{-1}$.

8. Two signed measures λ_1, λ_2 are said to be singular wrt each other if $|\lambda_1| \perp |\lambda_2|$. In the following μ is a measure.

   (a) Show that if ν is a signed measure such that $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$. (b) Show that if ν_1, ν_2 are singular wrt μ then so is $c_1\nu_1 + c_2\nu_2$. (c) Prove the uniqueness assumption in the Lebesgue decomposition theorem for two measures.

9. Extend the Radon-Nikodym theorem to the case of signed measures.

10. Use the following example to show that the hypothesis in the Radon-Nikodym theorem that μ is σ-finite cannot be omitted. Let $X = [0, 1], \mathcal{B}$ the class of Lebesgue measurable subsets of $[0, 1]$, ν be Lebesgue measure and μ be the counting measure on $\mathcal{B}$. Then ν is finite and absolutely continuous wrt μ but there is no function f such that $\nu(E) = \int_E f d\mu$ for all $E \in \mathcal{B}$.

11. **Complex measures:** A set function ν that assigns a complex number $\nu(E)$ to each E in a σ-algebra $\mathcal{B}$ is called a complex measure if $\nu(\phi) = 0$ and for each countable disjoint union $\cup E_i$ of sets in $\mathcal{B}$ we have $\nu(\cup E_i) = \sum \nu(E_i)$ with absolute convergence on the right.

    (a) Show that each complex measure ν can be expressed as $\nu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$, where $\mu_1, \mu_2, \mu_3, \mu_4$ are finite measures. (b) Show that for each complex measure ν there is a measure μ and a complex valued measurable function ϕ with $|\phi| = 1$ such that for each E in $\mathcal{B}$ we have $\nu(E) = \int_E \phi d\mu$. (Apply the Radon-Nikodym theorem to the measure μ_i wrt the measure $\mu_1 + \mu_2 + \mu_3 + \mu_4$.)