Percolation Theory

Homework 1

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Setup. We consider the d-dimensional lattice: $\mathbb{L}^d = (V, E)$, where $V = \mathbb{Z}^d$ is the set of vertices and $E = \{\{x, y\} : \|x - y\|_1 = 1\}$ is the set of edges. We consider the Bernoulli percolation on $\mathbb{L}^d$, where each edge is open with probability $p \in [0, 1]$ independently of the others.
Probability space. We consider the sample space $\Omega = \{0, 1\}^E$ with the σ-algebra $\mathcal{F}$ generated by the cylinder sets of the form

$$\{\omega \in \Omega \colon \omega(e_i) = \epsilon_i(e), e_i \in E, \epsilon_i \in \{0, 1\} \text{ and } 1 \le i \le n, n \ge 1\}$$

and the product probability measure $\mathbb{P}_p$ given by it's marginals: $p\delta_{\{1\}} + (1 - p)\delta_{\{0\}}$.

Problem 1. For $u, v \in V$, the event that there is an open path from u to v is denoted by $\{u \longleftrightarrow v\}$. Show that $\{u \longleftrightarrow v\} \in \mathcal{F}$.

Solution. Let $S(u, v)$ be the set of all open paths from u to v. We first show that $S(u, v)$ is countable. Let $N = \|u - v\|_1$. Now, for each $n \in \mathbb{N}$, the set of paths of length n, from u to v denoted by $S_n(u, v)$ is finite. Observe that, $S(u, v) = \bigcup_{n \ge N} S_n(u, v)$. Since each $S_n(u, v)$ is finite, $S(u, v)$ is countable. Now,

$$\{u \longleftrightarrow v\} = \bigcup_{\gamma \in S(u,v)} \{\omega \in \Omega \colon \omega(e) = 1 \text{ for all } e \in \gamma\}$$

is a countable union of cylinder sets, hence $\{u \longleftrightarrow v\} \in \mathcal{F}$. $\square$

We denote $\mathcal{C} := \mathcal{C}(0) = \{v \in V : v \longleftrightarrow 0\}$ to be the open cluster containing the origin.

Problem 2. Show that $\{\#\mathcal{C} \ge 10\} \in \mathcal{F}$.

Solution. We will show that $\{\#\mathcal{C} \ge n\} \in \mathcal{F}$ for all $n \in \mathbb{N}$. Observe that,

$$\{\#\mathcal{C} \ge n\} = \bigcup_{u_1, \ldots, u_n \in \mathbb{Z}^d} \bigcap_{i=1}^{n} \{u_i \longleftrightarrow 0\}.$$

By the previous problem, $\{u_i \longleftrightarrow 0\} \in \mathcal{F}$ for all i, hence $\{\#\mathcal{C} \ge n\} \in \mathcal{F}$. In particular, $\{\#\mathcal{C} \ge 10\} \in \mathcal{F}$. $\square$

Problem 3. Show that $\{\#\mathcal{C} < \infty\} \in \mathcal{F}$.

Solution. As argued in the previous problem, $\{\#\mathcal{C} \ge n\} \in \mathcal{F}$ for all $n \in \mathbb{N}$. So $\{\#\mathcal{C} = n\} = \{\#\mathcal{C} \ge n\} \setminus \{\#\mathcal{C} \ge n + 1\} \in \mathcal{F}$. Now, $\{\#\mathcal{C} < \infty\} = \bigcup_{n \ge 1} \{\#\mathcal{C} = n\}$ is a countable union of sets in $\mathcal{F}$, hence $\{\#\mathcal{C} < \infty\} \in \mathcal{F}$. $\square$

Define $\Theta(p) := \mathbb{P}(\#\mathcal{C} = \infty)$ to be the percolation probability. The critical probability of the lattice $\mathbb{L}^d$ is defined as

$$p_c(d) = p_c(\mathbb{L}^d) := \inf \{p \in [0, 1] \colon \Theta(p) > 0\}$$

Problem 4. Show that $p_c(1) = 1$.

Solution. For $d = 1$, note that there can be exactly n many open cluster of of size n. So,

$$\begin{aligned}\mathbb{P}_p(\exists \text{ a cluster of size } n) &\leq \mathbb{E}_p(\# \text{ of clusters of size } n) \\ &= np^n(1-p)^2 \leq np^n \to 0,\end{aligned}$$

as $n \to \infty$ and for $p < 1$. Thus, $p_c = 1$. $\square$