Percolation Theory

Homework 10

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MM2408

Problem 1. Fix N. Verify the hypotheses of the Subadditive Theorem for $\left\{X^N_{m,n} : 0 \le m \le n\right\}$, where

$$X^N_{m,n} := \max\left\{X_{m,n}, -N(n-m)\right\}.$$

In addition, show that $\alpha^N := \inf_{n\ge 1} \frac{1}{n}\mathbb{E}\left[\left(X^N_{0,n}\right)^+\right] \ge -N > -\infty$.

Solution. Note that $X^N_{0,0} = \max\{0, -N\cdot 0\} = 0$. Since for all $0 \le m \le n$, $X_{0,n} \le X_{0,m} + X_{m,n}$ holds, so does for the truncated version,

$$\begin{aligned}
X^N_{0,n} &\le \max\left\{X_{0,m} + X_{m,n}, -Nm - N(n-m)\right\}\\
&\le \max\left\{X_{0,m}, -Nm\right\} + \max\left\{X_{m,n}, -N(n-m)\right\}\\
&= X^N_{0,m} + X^N_{m,n}.
\end{aligned}$$

We now show that for all k, $\left\{X^N_{(n-1)k,nk} : n \ge 1\right\}$ is a stationary process. Fix k and define $Y_n := X^N_{(n-1)k,nk}$. For $x_1, x_2, \ldots, x_s$,

$$\begin{aligned}
&\mathbb{P}\left(X^N_{0,k} \le x_1, \ldots, X^N_{(s-1)k,sk} \le x_s\right)\\
&\quad= \mathbb{P}\left(-Nk \le X_{0,k} \le x_1, \ldots, -Nk \le X_{(s-1)k,sk} \le x_s\right)\\
&\quad= \mathbb{P}\left(-Nk \le X_{rk,(r+1)k} \le x_1, \ldots, -Nk \le X_{(r+s-1)k,(r+s)k} \le x_s\right)\\
&\quad= \mathbb{P}\left(X^N_{rk,(r+1)k} \le x_1, \ldots, X^N_{(r+s-1)k,(r+s)k} \le x_s\right),
\end{aligned}$$

where the second equality follows from the stationarity of $\left\{X_{(n-1)k,nk} : n \ge 1\right\}$. Since $x_1, x_2, \ldots, x_s$ and s were arbitrary, this shows that $\left\{X^N_{(n-1)k,nk} : n \ge 1\right\}$ is a stationary process.

Next we show that for every $m \ge 1$, $\left\{X^N_{m,m+k} : k \ge 0\right\} \overset{d}{=} \left\{X^N_{m+1,m+k+1} : k \ge 0\right\}$. Fix $m \ge 1$. Again for $x_1, x_2, \ldots, x_s$,

$$\begin{aligned}
&\mathbb{P}\left(X^N_{m,m+k} \le x_1, \ldots, X^N_{m+s,m+k+s} \le x_s\right)\\
&\quad= \mathbb{P}\left(-kN \le X_{m,m+k} \le x_1, \ldots, -kN \le X_{m+s,m+k+s} \le x_s\right)\\
&\quad= \mathbb{P}\left(-kN \le X_{m+1,m+k+1} \le x_1, \ldots, -kN \le X_{m+s+1,m+k+s+1} \le x_s\right)\\
&\quad= \mathbb{P}\left(X^N_{m+1,m+k+1} \le x_1, \ldots, X^N_{m+s+1,m+k+s+1} \le x_s\right),
\end{aligned}$$

where the second equality follows from $\{X_{m,m+k} : k \ge 0\} \overset{d}{=} \{X_{m+1,m+k+1} : k \ge 0\}$. Since again $x_1, x_2, \ldots, x_s$ and s were arbitrary, this shows our claim.

To show $\mathbb{E}\left[\left(X^N_{0,1}\right)^+\right] < \infty$, observe that $\left(X^N_{0,1}\right)^+ = \max\left\{X_{0,1}, -N\right\}^+ = \max\left\{X^+_{0,1}, 0\right\} = X^+_{0,1}$.

Now as

$$\mathbb{E}\left[X_{0,n}^N\right] \leq \mathbb{E}\left[X_{0,m}^N\right] + \mathbb{E}\left[X_{m,n}^N\right]$$
$$= \mathbb{E}\left[X_{0,m}^N\right] + \mathbb{E}\left[X_{0,n-m}^N\right]$$

for all $0 \leq m \leq n$. Writing $\alpha_k^N := \mathbb{E}\left[X_{0,k}^N\right]$, we get $\alpha_n^N \leq \alpha_m^N + \alpha_{n-m}^N$ for all $0 \leq m \leq n$. Thus by Fekete's Lemma,

$$\alpha^N = \lim_{n\to\infty} \frac{1}{n}\alpha_n^N = \inf_{n\geq 1} \frac{1}{n}\mathbb{E}\left[\left(X_{0,n}^N\right)^+\right] \geq -N,$$

where the last inequality follows from

$$\frac{X_{0,n}^N}{n} = \frac{\max\{X_{0,n}, -Nn\}}{n} \geq -N.$$

Hence $\alpha^N \geq -N$. □

Problem 2 (Fatou's Lemma)**.** Show the following version of Fatou's Lemma.

(i) If $\{X_n\}_{n\geq 1}$ is uniformly integrable from above (i.e., $\{X_n^+\}_{n\geq 1}$ is uniformly integrable), then

$$\mathbb{E}\left[\limsup_{n\to\infty} X_n\right] \geq \limsup_{n\to\infty} \mathbb{E}[X_n]$$

(ii) If $\{X_n\}_{n\geq 1}$ is uniformly integrable from below (i.e., $\{X_n^-\}_{n\geq 1}$ is uniformly integrable), then

$$\mathbb{E}\left[\liminf_{n\to\infty} X_n\right] \leq \liminf_{n\to\infty} \mathbb{E}[X_n]$$

Solution to (ii). Define $X := \liminf_{n\to\infty} X_n$. Let $\epsilon > 0$. Due to uniform integrability of $\{X_n^-\}_{n\geq 1}$, there exists $c > 0$ such that

$$\mathbb{E}\left[X_n^- \mathbb{1}_{\{X_n^- > c\}}\right] < \epsilon \quad \text{for all } n \geq 1.$$

Since

$$X + c \leq \liminf_{n\to\infty} (X_n + c)^+,$$

the monotonicty and the standard Fatou's Lemma imply

$$\mathbb{E}[X] + c \leq \mathbb{E}\left[\liminf_{n\to\infty} (X_n + c)^+\right]$$
$$\leq \liminf_{n\to\infty} \mathbb{E}\left[(X_n + c)^+\right].$$

As

$$(X_n + c)^+ = (X_n + c) + (X_n + c)^- \leq (X_n + c) + X_n^- \mathbb{1}_{\{X_n^- > c\}},$$

we have

$$\mathbb{E}\left[(X_n + c)^+\right] \leq \mathbb{E}[X_n + c] + \mathbb{E}\left[X_n^- \mathbb{1}_{\{X_n^- > c\}}\right] \leq \mathbb{E}[X_n] + c + \epsilon,$$

hence

$$\mathbb{E}[X] \leq \liminf_{n\to\infty} \mathbb{E}[X_n] + \epsilon.$$

This gives the asertion. □

Solution to (i). Taking $Y_n = -X_n$, and applying the result of (ii) to $\{Y_n\}_{n\geq 1}$, we get the desired result. □