Percolation Theory

Homework 11

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Problem 1. Consider the model of Oriented Percolation with parameter p. Define

$$\overline{\xi}_n := \left\{ m \in \mathbb{Z} \colon (m,n) \in \mathbb{Z}^2_{\text{even}} \& \exists\ (k,0) \in \mathbb{Z}^2_{\text{even}} \text{ with } k \le 0 \text{ such that } (k,0) \overset{\text{open}}{\longrightarrow} (m,n) \right\}.$$

Show that for $p > 0$, and for all $n \in \mathbb{N}$, $\overline{\xi}_n \neq \phi$ almost surely.

Solution. Fix $n \in \mathbb{N}$ and for $k \le 0$ take

$$\xi_{n,k} := \left\{ m \in \mathbb{Z} \colon (m,n) \in \mathbb{Z}^2_{\text{even}} \text{ such that } (k,0) \overset{\text{open}}{\longrightarrow} (m,n) \right\}.$$

Clearly $\overline{\xi}_n = \bigcup_{k \le 0} \xi_{n,k}$. Consider the sequence of events $A_k = \{\xi_{n,k} = \phi\}$ for $k = 0, -2n, -4n, \ldots$. Our construction ensures that, all n-length paths from $(k,0)$ to the line $y = n$ are formed by the edges

$$E_k = \left\{ \left(\overrightarrow{(x,y),(x+1,y+1)} \right), \left(\overrightarrow{(x,y),(x+1,y-1)} \right) \colon (x,y) \in \mathbb{Z}^2, x - y \le k \le x + y, y \le n - 1 \right\}.$$

Now the edges $\{E_{2rn} \colon r \le 0\}$ are disjoint, thus $\{A_{2rn}\}_{r \le 0}$ are independent. Note that $\mathbb{P}(A_{2rn}) = \mathbb{P}(A_0)$ for all $r \le 0$, due to translation invariance. Since,

$$\mathbb{P}(A_0^c) \ge \mathbb{P}(\text{ the path } ((0,0),(1,1),\ldots,(n,n)) \text{ is open}) = p^n > 0,$$

we have $\mathbb{P}(A_0) < 1$. So,

$$\begin{aligned}
\mathbb{P}(\overline{\xi}_n = \phi) &= \mathbb{P}(\xi_{n,k} = \phi \text{ for all } k \le 0) \\
&= \mathbb{P}\left(\bigcap_{k \le 0} A_k \right) \\
&\le \mathbb{P}\left(\bigcap_{r \le 0} A_{2rn} \right) \\
&= \lim_{k \to \infty} \mathbb{P}\left(\bigcap_{-k \le r \le 0} A_{2rn} \right) \\
&= \lim_{k \to \infty} \mathbb{P}(A_0)^{k+1} = 0
\end{aligned}$$

□

Problem 2. Define $\bar{l}_{n,m}$ for $0 \leq n \leq m$ for the setup of Oriented Percolation, and show that the Subadditive Ergodic Theorem holds for $\{\bar{l}_{n,m} \colon 0 \leq n \leq m\}$.

Solution. Define $\bar{\xi}_n^+ := \bar{\xi}_n^{\mathbb{Z}_{\geq 0}}$, $r_n^+ := r_n^{\mathbb{Z}_{\geq 0}}$ and $l_n^+ := l_n^{\mathbb{Z}_{\geq 0}}$. Imitating the solution of Problem 1, we may show for $p > 0$, $\bar{\xi}_n^+ \neq \phi$ a.s., thus we can conclude that $l_n^+ > -\infty$ almost surely. Now define $\{\bar{l}_{n,m} \colon 0 \leq n \leq m\}$ as

$$\bar{l}_{n,m} = \sup \left\{ l_n^+ - u \colon (u,m) \in \mathbb{Z}^2_{\text{even}} \text{ such that } \exists\, v \geq l_n^+ \text{ with } (v,n) \xrightarrow{\text{open}} (u,m) \right\}.$$

Now we show that $\{\bar{l}_{n,m} \colon 0 \leq n \leq m\}$ satisfies the hypotheses of the Subadditive Ergodic Theorem.

(i) By definition, $\bar{l}_{0,0} = 0$.

(ii) FTSOC, assume that ω is a configuration such that $\bar{l}_{0,m}(\omega) > \bar{l}_{0,n}(\omega) + \bar{l}_{n,m}(\omega)$ for some $0 \leq n \leq m$. Then any path from $(\bar{l}_{0,m}(\omega), m)$ to the ray $\mathbb{Z}_{\geq 0} \times \{0\}$ must pass through a path connecting $(\bar{l}_{0,n}(\omega), n)$ and $\mathbb{Z}_{\geq 0} \times \{0\}$ or the ray $(-\infty, \bar{l}_{0,n}(\omega)) \times \{n\}$. This contradicts the definition of $\bar{l}_{0,n}(\omega)$ and $\bar{l}_{n,m}(\omega)$. So we have,

$$\bar{l}_{0,m} \leq \bar{l}_{0,n} + \bar{l}_{n,m}.$$

for all $0 \leq n \leq m$.

(iii) Translation invariance ensures that for any fixed $k > 0$, $\{\bar{l}_{(n-1)k,nk}\}_{n \geq 0}$ is an i.i.d. (and thus stationary) sequence of random variables.

(iv) Again by translation invariance, for any fixed $m \geq 0$, $\{\bar{l}_{m,k} \colon k \geq 0\} \stackrel{d}{=} \{\bar{l}_{m+1,k+1} \colon k \geq 0\}$.

(v) Finally, we have $\bar{l}_{0,n} \leq 1$ and $\inf_{n \to \infty} l_n^+ = -\infty$.

□