Percolation Theory

Homework 12

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Problem 1. For the Poisson Boolean Model $(\mathbf{X}, \lambda, (\rho_i))$ in $\mathbb{R}^d$, with $\rho_i \overset{\text{iid}}{\sim} \rho$ where $\rho > 0$ a.s., show that

$$\mathbb{P}(\mathbf{0} \in C) = 1 - e^{-\lambda \pi_d \mathbb{E}[\rho^d]}$$

Here C denotes the covered volume of the said model.

Solution. **Case 1.** ρ is discrete.
Assume that ρ takes distinct values $\{r_i\}_{i \in I}$ with probabilities $\{p_i\}_{i \in I}$, where the indexing set I is finite or countably infinite, and $\sum_{i \in I} p_i = 1$. Consider the collection of independent Poisson Boolean Models $\{(X_i, \lambda p_i, r_i) \mid i \in I\}$.
Superposing these Poisson Boolean models and using the additivity property, we get back the $(\mathbf{X}, \lambda, \rho)$ model. Let $\{C_i \mid i \in I\}$ denote the covered region in the $(X_i, \lambda p_i, r_i)$ model. Then $C = \bigcup_{i \in I} C_i$ and

$$\begin{aligned}
\mathbb{P}_\lambda(\mathbf{0} \notin C) &= \mathbb{P}\left(\bigcap_{i \in I} \{\mathbf{0} \notin C_i\}\right) \\
&= \prod_{i \in I} \mathbb{P}_{\lambda p_i}(\mathbf{0} \notin C_i) \\
&= \prod_{i \in I} \mathbb{P}_{\lambda p_i}(\text{No point of } X_i \text{ lies in } B(\mathbf{0}, r_i)) \\
&= \prod_{i \in I} e^{-\lambda p_i \pi_d r_i^d} \\
&= e^{-\lambda \pi_d \sum_{i \in I} p_i r_i^d} = e^{-\lambda \pi_d \mathbb{E}[\rho^d]}
\end{aligned}$$

So,

$$\mathbb{P}_\lambda(\mathbf{0} \in C) \quad = 1 - e^{-\lambda \pi_d \mathbb{E}[\rho^d]}$$

Case 2. ρ is not discrete (measurable).
Let ρ be defined on a measure space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\rho : \Omega \to \mathbb{R}$ is a measurable function. There exists an increasing sequence of functions $\rho_n : \Omega \to \mathbb{R}$ (random variables) such that $0 \le \rho_1 \le \rho_2 \le \dots$ and $\rho_n \xrightarrow{\text{a.s.}} \rho$.
Consider a Poisson Point process $(\mathbf{X}, \lambda)$ in $\mathbb{R}^d$ defined on the space $(\Omega', \mathcal{F}', \mathbb{P}')$ where $\mathbf{X}$ is countable $\mathbb{P}'$-almost surely for $\lambda > 0$.
Let $\left(\Omega' \times \Omega^{\mathbb{N}}, \mathcal{F}' \otimes (\otimes_{\mathbb{N}} \mathcal{F}), \mathbb{P}\right)$ denote the product of $(\Omega', \mathcal{F}', \mathbb{P}')$ and a countable collection of $(\Omega, \mathcal{F}, \mathbb{P})$, where $\tilde{\mathcal{F}} := \mathcal{F}' \otimes (\otimes_{\mathbb{N}} \mathcal{F})$ is the σ-algebra generated by the cylinder sets.

Then, for all $(\mathbf{x},(w_i)) \in \tilde{\Omega} := \Omega' \times \Omega^{\mathbb{N}}$, we define covered regions

$$C_j(\mathbf{x},(w_i)) = \bigcup_{i=1}^{\infty} B(x_i, \rho_j(w_i)) \tag{1}$$

for every $j \in \mathbb{N}$.
Note that here $\Omega' = \left(\mathbb{R}^d\right)^{\mathbb{N}}$ and $\mathbf{x} = (x_1, x_2, \ldots)$. The random objects C_j are functions $C_j : \tilde{\Omega} \longrightarrow \mathcal{B}(\mathbb{R}^d)$ such that the map $\ell \circ C_j : \tilde{\Omega} \longrightarrow [0,\infty]$ is measurable (here ℓ is the Lebesgue measure).
Let C be the covered region corresponding to ρ in $\mathbb{R}^d$, defined as in (1). As $\rho_1 \leq \rho_2 \leq \ldots$ on Ω, we have

$$C_j(\mathbf{x},(w_i)) \subseteq C_{j+1}(\mathbf{x},(w_i))$$

for all $(w_i) \in \Omega^{\mathbb{N}}$.
Thus, as random objects on $\Omega^{\mathbb{N}}$, $C_j(\mathbf{x}, 0) \subseteq C_{j+1}(\mathbf{x}, \cdot)$ for all $\mathbf{x} \in \Omega'$. Also, as $\lim_{n\to\infty} \rho_n = \rho$ almost everywhere on Ω, we have

$$\begin{aligned}
\bigcup_{j=1}^{\infty} C_j(\mathbf{x},(\omega_i)) &= \bigcup_{j=1}^{\infty}\bigcup_{i=1}^{\infty} B(x_i, \rho_j(\omega_i)) \\
&= \bigcup_{i=1}^{\infty}\left(\bigcup_{j=1}^{\infty} B(x_i, \rho_j(\omega_i))\right) \\
&= \bigcup_{i=1}^{\infty} B\left(x_i, \lim_{j\to\infty} \rho_j(\omega_i)\right) = C(\mathbf{x},(w_i))_i
\end{aligned}$$

for almost all $(w_i) \in \Omega^{\mathbb{N}}$ and every $\mathbf{x} \in \Omega'$. Therefore, $\bigcup_{j=1}^{\infty} C_j = C$ almost surely on $\tilde{\Omega}$.. Now,

$$\begin{aligned}
\mathbb{P}_\lambda(\mathbf{0} \notin C) &= \mathbb{P}_\lambda(\mathbf{0} \notin C) \\
&= \mathbb{P}_\lambda\left(\mathbf{0} \notin \bigcup_{j=1}^{\infty} C_j\right) \\
&= \mathbb{P}_\lambda\left(\bigcap_{j=1}^{\infty} \{\mathbf{0} \notin C_j\}\right) \\
&= \lim_{n\to\infty} \mathbb{P}_\lambda(\mathbf{0} \notin C_n) \\
&\overset{\text{Case 1.}}{=} \lim_{n\to\infty} e^{-\lambda\pi_d \mathbb{E}\left[\rho_n^d\right]} \\
&\overset{\text{MCT}}{=} e^{-\lambda\pi_d \lim_{n\to\infty} \mathbb{E}\left[\rho_n^d\right]} = e^{-\lambda\pi_d \mathbb{E}\left[\rho^d\right]}
\end{aligned}$$

Therefore, $\mathbb{P}_\lambda(\mathbf{0} \in C) = 1 - e^{-\lambda\pi_d \mathbb{E}\left[\rho^d\right]}$.
Since all random variables ρ_n were positive almost surely, this also works even if $\mathbb{E}\left[\rho^d\right] = \infty$. In that case, we have $\mathbb{P}_\lambda(\mathbf{0} \in C) = 1$ and due to translation invariance,

$$\mathbb{P}_\lambda(\mathbf{u} \in C) = 1$$

for all $\mathbf{u} \in \mathbb{R}^d$. □