Percolation Theory

Homework 2

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Binary tree. We consider a rooted binary tree $\mathbb{T} := (V, E)$ with the vertices $V = \bigcup_{n \geq 0} \{0,1\}^n$, where $\{0,1\}^0 := \{\phi\}$ is the root and the edges $E = \{\{x, a(x)\} : x \in V \setminus \{\phi\}\}$, where $a(x) := (x_1, \ldots, x_{n-1}) \in \{0,1\}^{n-1}$ for each $x \in \{0,1\}^n$.

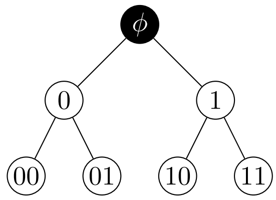

FIGURE 1. Binary tree $\mathbb{T}$

We take the same percolation structure as the Bernoulli percolation on $\mathbb{L}^d$, where each edge is open with probability $p \in [0,1]$ independently of the others.

Denote $C_0 := \{x \in \mathbb{T} \colon x \longleftrightarrow \phi\}$. Then $\{\#C_0 = \infty\}$ is the event that the root ϕ is contained in an infinite cluster.

Problem 1. Find $p_c(\mathbb{T})$.

Solution. To conclude $p_c(\mathbb{T}) = 1/2$, we will show that,

$$\Theta(p) = \begin{cases} 0 & \text{if } p \leq 1/2, \\ \frac{2p-1}{p^2} & \text{if } p > 1/2. \end{cases}$$

Let A_0 be the event that node 0 is in an infinite component "going forward", that is, not including ϕ. Let A_1 be the same for 1. By self-similarity of $\mathbb{T}$, we have

$$\mathbb{P}_p(A_0) = \mathbb{P}_p(A_1) = \Theta(p)$$

Let $B_i, i = 0,1$ be the event that there is an edge from ϕ to i and that A_i holds. So, $B_i = \{\phi \longleftrightarrow i\} \cap A_i, i = 0,1$. Then,

$$\mathbb{P}_p(B_i) = p\Theta(p).$$

As B_0 and B_1 are independent,

$$\begin{aligned} \Theta(p) &= \mathbb{P}_p(B_0 \cup B_1) \\ &= \mathbb{P}_p(B_0) + \mathbb{P}_p(B_1) - \mathbb{P}_p(B_0 \cap B_1) \\ &= 2p\Theta(p) - (p\Theta(p))^2. \end{aligned} \tag{1}$$

One solution of this equation is $\Theta(p) = 0$. If $\Theta(p) \neq 0$, we can divide by $\Theta(p)$ to get

$$\Theta(p) = \frac{2p-1}{p^2}.$$

So, for $p > \frac{1}{2}$, $\frac{2p-1}{p^2}$ is a solution. Note that, $\Theta(p) = 0$ for all $p < \frac{1}{2}$, as otherwise (1) would require $\Theta(p) < 0$, which is absured. When $p = \frac{1}{2}$, (1) has only solution $\Theta(p) = 0$. To show that $\Theta(p) = \frac{2p-1}{p^2}$ is the only solution for $p > \frac{1}{2}$, we turn to the second moment method.

For each $n \geq 1$, define $X_n := \sum_{x \in \{0,1\}^n} \mathbb{1}[x \in C_0]$. Then,

$$\mathbb{E}[X_n] = \sum_{x \in \{0,1\}^n} \mathbb{P}(x \in C_0) = 2^n p^n,$$

as there are exactly 2^n vertices at level n (i.e., $\{0,1\}^n$) and each of them is connected to the root ϕ has probability p^n. Now,

$$\begin{aligned}
\mathbb{E}[X_n^2] &= \sum_{\substack{x,y \in \{0,1\}^n \\ x \neq y}} \mathbb{P}(x \in C_0, y \in C_0) + \sum_{x \in \{0,1\}^n} \mathbb{P}(x \in C_0) \\
&= \sum_{r=0}^{n-1} 2^n \cdot 2^{n-r-1} p^{2n-r} + (2p)^n \\
&\leq (2p)^{2n} \left(\frac{p}{2p-1} + (2p)^{-n} \right),
\end{aligned}$$

where the first term in the second equality follows form the following counting argument: since $\mathbb{T}$ is a tree, for $x \neq y$ in $\{0,1\}^n$, there exists an unique $c \equiv c(x,y,0) \in \{0,1\}^r$ with $r < n$ such that c lies on the path from x to 0 and y to x. So there must be precisely $2n - r$ many open edges for the occurence of $\{x \in C_0, y \in C_0\}$, which yields, $\mathbb{P}(x \in C_0, y \in C_0) = p^{2n-r}$. To count the number of such paths, note that, $c(x,y,0) = a^{n-r}(x)$ has exactly 2^{n-r} many descendents in $\{0,1\}^n$ among which 2^{n-r-1} are of $a^{n-r-1}(x)$. So, for each $x \in \{0,1\}^n$, there are exactly $2^{n-r} - 2^{n-r-1} = 2^{n-r-1}$ many choices for $y \in \{0,1\}^n$ with $c(x,y,0) = a^{n-r}(x)$.
Applying the second moment method,

$$\mathbb{P}(X_n > 0) \geq \frac{\mathbb{E}[X_n]^2}{\mathbb{E}[X_n^2]} \geq \frac{1}{\frac{p}{2p-1} + (2p)^{-n}}.$$

Notice that, $\{X_1 > 0\} \supseteq \{X_2 > 0\} \supseteq \cdots \supseteq \{X_n > 0\} \supseteq \cdots$ is a decreasing sequence of events converging to $\cap_{n \geq 1}\{X_n > 0\} \subseteq \{\#C_0 = \infty\}$. Then Monotone convergence theorem yields,

$$\begin{aligned}
\mathbb{P}(\#C_0 = \infty) &\geq \mathbb{P}(\cap_{n \geq 1}\{X_n > 0\}) \\
&= \lim_{n \to \infty} \mathbb{P}(X_n > 0) \\
&= \lim_{n \to \infty} \frac{1}{\frac{p}{2p-1} + (2p)^{-n}} \\
&= \frac{2p-1}{p}.
\end{aligned}$$

which is strictly positive for $p > \frac{1}{2}$. This shows our claim and hence completes the proof. $\square$

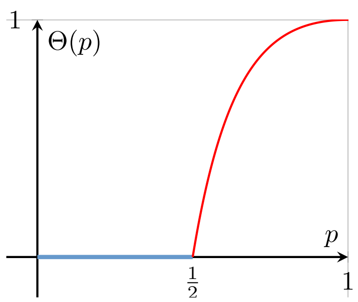

FIGURE 2. Plot of $\Theta(p)$

Problem 2. Using Kolmogorov's $0-1$ law, show that if $\Theta(p) > 0$, then

$$\mathbb{P}_p\left(\exists\ u \in V(\mathbb{L}^d) \text{ such that } \#\mathcal{C}(u) = \infty\right) = 1.$$

Solution. Let $e_1, e_2, \ldots$ be an enumeration of the edges of $\mathbb{L}^d$. Let A_n be the event that $C = \{e_n \text{ is open}\} = \{\omega \in \Omega \colon \omega(e_n) = 1\}$. Then $\{A_n\}_{n\geq 1}$ is a sequence of independent events. Let

$$\tau = \bigcap_{n\geq 1} \bigcup_{k\geq n} \sigma(A_1, \ldots, A_k)$$

be the tail σ-field. Call the event $E \coloneqq \left\{\exists\ u \in V(\mathbb{L}^d) \text{ such that } \#\mathcal{C}(u) = \infty\right\} = \bigcup_{u\in V}\{\#\mathcal{C}(u) = \infty\}$. Let F be a finite subset of edges. For a cluster to be infinite, it must intersect $\cap_{e\in F^c} A_e$. And changing the state of any edge within or near F can only affect the cluster within or near F. It cannot affect whether an infinite cluster exists entirely in F^c. Thus E remains unchanged if the states within F are altered, so C is independent of $\sigma(A_e : e \in F)$. Since the choice of F is arbitrary, and configuration of any finite F does not affect the occurence of E, we take $F_k \coloneqq \{e_1, \ldots, e_k\}$. This gives that, $E \in \tau$. By Kolmogorov's $0-1$ law, $\mathbb{P}(E) = 0$ or 1. Since $\Theta(p) > 0$ and $\{\#C(0) = \infty\} \subseteq E$ we have $\mathbb{P}(E) = 1$. □

Problem 3. Fix $u, v \in V(\mathbb{L}^d)$. Show that $\{u \longleftrightarrow v\}$ is an increasing event.

Solution. Since, a new edge being open can only increase the number of paths between two vertices, the event $\{u \longleftrightarrow v\}$ is increasing. To make it more explicit, let $\omega \in \{u \longleftrightarrow v\}$, then there exists an open path γ containing edges $e_1, \ldots, e_n$ from u to v in ω, i.e., $\omega(e_i) = 1$ for all $i = 1, \ldots, n$. Let $\omega' \in \Omega$ be such that $\omega \preceq \omega'$. Then $\omega'(e_i) = 1$ for all $i = 1, \ldots, n$. Thus, γ is also an open path from u to v in ω', so $\omega' \in \{u \longleftrightarrow v\}$. Hence, $\{u \longleftrightarrow v\}$ is an increasing event. □