Percolation Theory

Homework 3

BIKRAM HALDER
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Problem 1. For an increasing event A, and $0 \le p \le p' \le 1$ show that

$$\mathbb{P}_p(A) \le \mathbb{P}_{p'}(A).$$

Solution. We use coupling argument as done in the class. Let $\{U_e : e \in E(\mathbb{L}^d)\}$ be a collection i.i.d. $\mathsf{Unif}[0,1]$ random variables on some probability space $(\Xi, \mathcal{G}, \mathbb{P})$. We consider two copies of lattice $\mathbb{L}^d$ as L and L'. We fix $0 \le p \le p' \le 1$, and define two collections of labels (open/closed) $\{X_e : e \in E(L)\}$ and $\{Y_e : e \in E(L')\}$ as follows:

$$X_e = \begin{cases} 1 & \text{if } U_e < p, \\ 0 & \text{if } U_e \ge p, \end{cases} \qquad Y_e = \begin{cases} 1 & \text{if } U_e < p', \\ 0 & \text{if } U_e \ge p'. \end{cases}$$

Clearly, the induced measure on L is $\mathbb{P}_p$ and on L' is $\mathbb{P}_{p'}$. Now note that, if $X_e = 1$ then $Y_e = 1$. For an increasing event A, let $\tilde{A}$ and $\tilde{A}'$ be the corresponding events in L and L' respectively. Then, we have $\tilde{A} \subseteq \tilde{A}'$ and hence

$$\mathbb{P}_p(A) = \mathbb{P}(\tilde{A}) \le \mathbb{P}(\tilde{A}') = \mathbb{P}_{p'}(A).$$

□

Problem 2. Show that, if A is an increasing event, then A^c is a decreasing event.

Solution. Let $\omega \in A^c$ then $\omega \notin A$. As A is an increasing event, for any $\omega' \preceq \omega$, we have $\omega' \notin A$, otherwise if $\omega' \in A$, then $\omega \in A$ which is a contradiction. Hence, $\omega' \notin A$. Therefore, $\omega' \in A^c$. Therefore, A^c is a decreasing event. □