Percolation Theory

Homework 4

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For an event $A \subseteq \{0,1\}^n$ and $t \in \{0,1\}$, define

$$A_t = \{x \in \{0,1\}^{n-1} : (x,t) \in A\}$$

and the probability measure on $\{0,1\}^n$ is P_n which is product of n many $\mathsf{Ber}(p)$ measures with marginals $p\delta_{\{1\}} + (1-p)\delta_{\{0\}}$.

Problem 1. Let A be an increasing event. Show that A_0 and A_1 are also increasing and $A_0 \subseteq A_1$.

Solution. Let $\omega \in A_0$ then $(\omega, 0) \in A$. Since A is increasing, $(\omega, 1) \in A$, so $\omega \in A_1$. Thus, $A_0 \subseteq A_1$. Now, let $\omega \in A_0$ and $\omega' \in \{0,1\}^{n-1}$ such that $\omega \preceq \omega'$. Then $(\omega, 0) \preceq (\omega', 0)$ and $(\omega', 0) \in A$. Thus, $\omega' \in A_0$. Therefore, A_0 is increasing. Similarly, A_1 is increasing.

Now let $\omega \in A_0$. Then $(\omega, 0) \in A$. Since A is increasing, $(\omega, 1) \in A$. Thus, $\omega \in A_1$. Therefore, $A_0 \subseteq A_1$. $\square$

Lemma 1. *For increasing events A and B,*

$$A \square B = \{a + b \colon a \in A, b \in B \text{ and } a_i b_i = 0 \ \forall\ i \in [n]\}.$$

Proof. Done in class. $\square$

Problem 2. Let A and B be both increasing events and take $C = A \square B$. Use Lemma 1 to show the following:

(a) $C_0 = A_0 \square B_0$ and $C_1 = (A_0 \square B_1) \cup (A_1 \square B_0)$.
(b) Use Problem 1 to show that $C_0 \subseteq (A_0 \square B_1) \cap (A_1 \square B_0)$ and $C_1 \subseteq A_1 \square B_1$.

Solution to (a). Let $x \in C_0$. Then $(x, 0) \in C$. Since $C = A \square B$, there exists $a \in A$ and $b \in B$ such that $(x, 0) = a + b$ and $a_i b_i = 0$ for all $i \in [n]$. In particular, $a_n = 0 = b_n$ or $a_n = 1 = b_n$, but the latter case yields $a_n b_n = 1$, which can't happen. Thus, $x \in A_0 \square B_0$. Therefore, $C_0 \subseteq A_0 \square B_0$.
Again let $x \in A_0 \square B_0$. Then there exists $a \in A_0$ and $b \in B_0$ such that $x = a + b$ and $a_i b_i = 0$ for all $i \in [n]$. In particular, $a_n = 0 = b_n$. Thus, $(x, 0) = (a, 0) + (b, 0) = a + b$. Therefore, $x \in C_0$. Therefore, $A_0 \square B_0 \subseteq C_0$. Thus, $C_0 = A_0 \square B_0$.

Similarly, let $x \in C_1$. Then $(x, 1) \in C$. Since $C = A \square B$, there exists $a \in A$ and $b \in B$ such that $(x, 1) = a + b$ and $a_i b_i = 0$ for all $i \in [n]$. In particular, $a_n = 1$ and $b_n = 0$ or $a_n = 0$ and $b_n = 1$. Thus, either $a \in A_0$ and $b \in B_1$ or $a \in A_1$ and $b \in B_0$. Therefore, $x \in (A_0 \square B_1) \cup (A_1 \square B_0)$. Therefore, $C_1 \subseteq (A_0 \square B_1) \cup (A_1 \square B_0)$.
Again let $x \in (A_0 \square B_1) \cup (A_1 \square B_0)$. Then there exists $a \in A_0$ and $b \in B_1$ or $a \in A_1$ and $b \in B_0$ such that $x = a + b$ and $a_i b_i = 0$ for all $i \in [n]$. In particular, $a_n = 0$ and $b_n = 1$ or $a_n = 1$ and

$b_n = 0$. Thus, $(x, 1) = (a, 0) + (b, 1) = a + b$ or $(x, 1) = (a, 1) + (b, 0) = a + b$. Therefore, $x \in C_1$. Therefore, $(A_0 \square B_1) \cup (A_1 \square B_0) \subseteq C_1$. Thus, $C_1 = (A_0 \square B_1) \cup (A_1 \square B_0)$.

$\square$

Solution to (b). Let $\omega \in C_0 = A_0 \square B_0$. So, there exists $a \in A_0$ and $b \in B_0$ such that $\omega = a + b$ with $a_i b_i = 0$ for all $i \in [n-1]$ and $(a, 0) \in A, (b, 0) \in B$. Since A and B are increasing, by Problem 1 we have $A_0 \subseteq A_1$ and $B_0 \subseteq B_1$ and all of them are increasing. Therefore, $\omega = a + b$ with $a_i b_i = 0$ for all $i \in [n-1]$ and $(a, 1) \in A, (b, 0) \in B$. Thus, $\omega \in (A_0 \square B_1)$. Also, $\omega = a + b$ with $a_i b_i = 0$ for all $i \in [n-1]$ and $(a, 0) \in A, (b, 1) \in B$. Thus, $\omega \in (A_1 \square B_0)$. Therefore, $\omega \in (A_0 \square B_1) \cap (A_1 \square B_0)$. Therefore, $C_0 \subseteq (A_0 \square B_1) \cap (A_1 \square B_0)$.

Take $\omega \in C_1 = (A_0 \square B_1) \cup (A_1 \square B_0)$. Then there exists $a \in A_0$ and $b \in B_1$ or $a \in A_1$ and $b \in B_0$ such that $\omega = a + b$ with $a_i b_i = 0$ for all $i \in [n-1]$ and $(a, 0) \in A, (b, 1) \in B$ or $(a, 1) \in A, (b, 0) \in B$. Since A and B are increasing, by Problem 1 we have $A_0 \subseteq A_1$ and $B_0 \subseteq B_1$ and all of them are increasing. Therefore, $\omega = a + b$ with $a_i b_i = 0$ for all $i \in [n-1]$ and $(a, 1) \in A, (b, 1) \in B$. Thus, $\omega \in A_1 \square B_1$. Therefore, $C_1 \subseteq A_1 \square B_1$. $\square$

Problem 3. For any event A show that

$$P_n(A) = (1-p)P_{n-1}(A_0) + pP_{n-1}(A_1).$$

Solution. Let X_n be a $\mathsf{Ber}(p)$-valued random variable which determines the n-th bit of $\omega \in \{0, 1\}^n$. Then

$$\begin{aligned} P_n(A) &= \sum_{\omega \in A} P_n(\omega) = \sum_{\omega \in A} P_n(\omega_1, \dots, \omega_n) = \sum_{\omega \in A} P_{n-1}(\omega_1, \dots, \omega_{n-1}) P_1(\omega_n) \\ &= \sum_{\omega \in A_0} P_{n-1}(\omega_1, \dots, \omega_{n-1}) \mathbb{P}(X_n = 0) + \sum_{\omega \in A_1} P_{n-1}(\omega_1, \dots, \omega_{n-1}) \mathbb{P}(X_n = 1) \\ &= (1-p)P_{n-1}(A_0) + pP_{n-1}(A_1). \end{aligned}$$

$\square$