Percolation Theory

Homework 5

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For $n \geq 1$, define the box $B_n := [-n, n]^d \cap \mathbb{Z}^d$, boundary of B_n as $\partial B_n := \{v \in V : \|v\|_\infty = n\}$ and

$$\beta_n := \mathbb{P}(\mathcal{C} \cap \partial B_n \neq \emptyset).$$

Lemma 1. *For all $m, n \geq 1$, we have*

(i) $\beta_{m+n} \leq \#(\partial B_m)\beta_m\beta_n$,
(ii) $\beta_{m+n} \geq \frac{1}{2d\,\#(\partial B_m)}\beta_m\beta_n$.

Problem 1. Do similar computations as in Lecture 6 by taking the sequence $\{b_k\}_{k\geq 1}$ defined by

$$b_k := g_k - \log \beta_k + (d-1)\log 2$$

and show for all $m, n \geq 1$ that,

(i) (Subadditivity) $b_{m+n} \leq b_m + b_n$,
(ii) $g_n - \log \beta_n + (d-1)\log 2 \geq n\phi(p)$,
(iii) $\log \beta_n \leq -n\phi(p) + (d-1)\log n + c_2$, where c_2 is some constant,
(iv) $\beta_n \leq C_2 e^{-n\phi(p)} n^{d-1}$, where C_2 is some constant.

Solution. We have

$$\begin{aligned}\#(\partial B_m) &\leq 2d(2m+1)^{d-1} \\ &= 2d\left(2 + \frac{1}{m}\right)^{d-1} m^{d-1} \\ &\leq d\,3^d m^{d-1}.\end{aligned}$$

Also trivially,

$$\#(\partial B_m) \leq 2d\,\#(\partial B_m) \leq d^2 3^{d+1} m^{d-1}.$$

Then from Lemma 1, taking logarithm on both sides, we get

$$\log \beta_{m+n} \leq \log \beta_m + \log \beta_n + \log\left(d^2 3^{d+1} m^{d-1}\right), \tag{1}$$

$$\log \beta_{m+n} \geq \log \beta_m + \log \beta_n - \underbrace{\log\left(d^2 3^{d+1} m^{d-1}\right)}_{\text{Call it } g_m}. \tag{2}$$

Without loss of generality, we assume that $m \leq n$. Subtracting g_n from both sides of (2), and rearranging,

$$\begin{aligned} g_n - \log \beta_{m+n} &\leq g_n + g_m - \log \beta_m - \log \beta_n \\ &= (g_n - \log \beta_n) + (g_m - \log \beta_m) \end{aligned} \tag{3}$$

Note that $g_n = 2\log d + (d+1)\log 3 + (d-1)\log n$. So,

$$g_{m+n} - g_n = (d-1)\log\frac{m+n}{n} \leq (d-1)\log 2. \tag{4}$$

Now (3) and (4) gives

$$\begin{aligned} g_{m+n} - \log \beta_{m+n} &= g_{m+n} - g_n + g_n - \log \beta_{m+n} \\ &\leq (d-1)\log 2 + (g_n - \log \beta_n) + (g_m - \log \beta_m). \end{aligned} \tag{5}$$

Adding $(d-1)\log 2$ on both sides of (5), we get

$$g_{m+n} - \log \beta_{m+n} + (d-1)\log 2 \leq (g_n - \log \beta_n + (d-1)\log 2) + (g_m - \log \beta_m + (d-1)\log 2).$$

In other words, $b_{m+n} \leq b_m + b_n$. This proves the subadditivity of b_k (part (i)). By Fekete's lemma, the following limit exists and is given by

$$\lim_{n\to\infty} \frac{b_n}{n} = \inf_{k\geq 1} \frac{b_k}{k}. \tag{6}$$

Notice that $g_n + (d-1)\log 2 = o(n)$. So, from (5), we have

$$\underbrace{-\lim_{n\to\infty} \frac{\log \beta_n}{n}}_{\text{Call it } \phi(p)} = \inf_{k\geq 1} \frac{b_k}{k}. \tag{7}$$

So, $\phi(p) \leq \frac{b_n}{n}$ for all $n \geq 1$. In other words, for all $n \geq 1$,

$$g_n - \log \beta_n + (d-1)\log 2 \geq n\phi(p). \tag{8}$$

This proves part (ii). Rearranging (8),

$$\begin{aligned} \log \beta_n &\leq -n\phi(p) + g_n + (d-1)\log 2 \\ &= -n\phi(p) + (2\log d + (d+1)\log 3 + (d-1)\log n) + (d-1)\log 2 \\ &= -n\phi(p) + (d-1)\log n + c_2, \end{aligned} \tag{9}$$

where $c_2 = 2\log d + (d+1)\log 3 + (d-1)\log 2$ (constant). This proves part (iii). Finally, taking exponential on both sides of (9), we get

$$\begin{aligned} \beta_n &\leq e^{-n\phi(p)} n^{d-1} e^{c_2} \\ &\leq C_2 e^{-n\phi(p)} n^{d-1}, \end{aligned} \tag{10}$$

where $C_2 = e^{c_2}$ (constant). This proves part (iv). □

Problem 2. For $x \geq y \geq 0$ and $\gamma \in [1, +\infty)$ show that

$$x^\gamma p^\gamma + y^\gamma (1-p^\gamma) \leq (xp + y(1-p))^\gamma.$$

Solution. For $\gamma = 1$ or $p = 0$ or $x = y$ or one of x or y is zero, the inequality becomes an equality. So we assume that $p \in (0, 1]$, $\gamma > 1$ and $x \geq y > 0$, so $t := \frac{x}{y} \geq 1$, i.e., $t - 1 \geq 0$. Thus we are required to show that

$$t^\gamma p^\gamma + 1 - p^\gamma \leq (tp + (1-p))^\gamma.$$

Or equivalently,

$$((t-1)p + 1)^\gamma - (t^\gamma - 1)p^\gamma - 1 \geq 0.$$

Let $f(t) := ((t-1)p+1)^\gamma - (t^\gamma - 1)p^\gamma - 1$ for $t \geq 1$. Then $f(1) = 0$ and

$$\begin{aligned} f'(t) &= \gamma((t-1)p+1)^{\gamma-1} p - \gamma t^{\gamma-1} p^\gamma \\ &= \left(\left((t-1) + \frac{1}{p} \right)^{\gamma-1} - t^{\gamma-1} \right) \gamma p^\gamma \end{aligned}$$

$$\geq \left(((t-1)+1)^{\gamma-1} - t^{\gamma-1} \right) \gamma p^{\gamma} = 0,$$

where the last inequality follows from $1/p \geq 1$. Therefore, f is increasing in $[1, +\infty)$. So, for all $t \geq 1$, we have $f(t) \geq f(1) = 0$. This proves the inequality. $\square$