Percolation Theory

Homework 7

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We define

$$p_T := \inf \{p \in [0,1] \colon \mathbb{E}_p[\#\mathcal{C}(0)] = \infty\}.$$

We recall the following result from Homework 5.

Lemma 1.

$$\beta_n(p) := \mathbb{P}_p(\mathcal{C}(0) \cap \partial B_n \neq \emptyset) \leq C_d e^{-n\phi(p)} n^{d-1}$$

Problem 1. Show that $p_T = p_c$.

Solution. For $p \in [0,1]$, if $\mathbb{P}_p(\#\mathcal{C}(0) = \infty) > 0$ then we must have $\mathbb{E}_p[\#\mathcal{C}(0)] = \infty$. Therefore, $p_T \leq p_c$. Assume for the sake of contradiction that $p_T < p_c$. Then there exists a $p \in (p_T, p_c)$ so that $\mathbb{E}_p[\#\mathcal{C}(0)] = \infty$ and $\mathbb{P}_p(\#\mathcal{C}(0) = \infty) > 0$. Note that $\phi(p) > 0$ as $p < p_c$. Also $\mathbb{P}_p\left(\#\mathcal{C}(0) \geq (2n+1)^d\right) \leq \beta_n(p)$ and both $\{\mathbb{P}_p(\#\mathcal{C}(0) \geq k)\}_{k \geq 1}$ and $\{\beta_n(p)\}_{n \geq 1}$ are monotonically decreasing sequences. Therefore, we have

$$\begin{aligned}
\mathbb{E}_p[\#\mathcal{C}(0)] &= \sum_{k=1}^{\infty} \mathbb{P}_p(\#\mathcal{C}(0) \geq k) \\
&\leq \sum_{n=0}^{\infty} \left((2n+3)^d - (2n+1)^d\right) \mathbb{P}_p\left(\#\mathcal{C}(0) \geq (2n+1)^d\right) \\
&\leq \sum_{n=0}^{\infty} \left((2n+3)^d - (2n+1)^d\right) \beta_n(p) \\
&\leq C_d \sum_{n=0}^{\infty} \left((2n+3)^d - (2n+1)^d\right) n^{d-1} e^{-n\phi(p)},
\end{aligned}$$

where the third line follows form Lemma 1. As exponential term dominates any arbitary polynomial, for large enough n, we can have

$$\left((2n+3)^d - (2n+1)^d\right) n^{d-1} e^{-n\phi(p)} \leq 1/n^2,$$

ensuring the convergence and finiteness of the last sum in the above. This is a contradiction. Therefore, $p_T = p_c$. □