Percolation Theory

Homework 8

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For a nonnegetive function $f\colon [0,\infty) \to [0,\infty)$, we define

$$G_f := \big\{(x, y) \in \mathbb{Z}^2 \colon 0 \le y \le f(x)\big\},$$

and the critical probability for G_f as

$$p_c := \sup\{p \in [0, 1] \colon \mathbb{P}_p\,(G_f \text{ admits an unbounded open component}) = 0\}.$$

Problem 1. Using $0-1$ law show that, if $p > p_c(G_f)$, then there exists an unbounded open component with probability 1.

Solution. We will mostly imitate the solution of Problem 2 form Homework 2. Let $e_1, e_2, \ldots$ be an enumeration of the edges $E(G_f)$ (countable as G_f is a subgraph of $\mathbb{L}^d$). Let A_n be the event that $C = \{e_n \text{ is open}\} = \{\omega \in \Omega \colon \omega(e_n) = 1\}$. Then $\{A_n\}_{n\ge1}$ is a sequence of independent events which is inherited from the independence of $\{A_e\}_{e\in E(\mathbb{Z}^2)}$. Let

$$\tau = \bigcap_{n\ge1} \bigcup_{k\ge n} \sigma(A_1, \ldots, A_k)$$

be the tail σ-field. Call the event $E := \{\exists\ u \in V(G_f) \text{ such that } \#\mathcal{C}(u) = \infty\}$. Let F be a finite subset of edges. For a cluster to be infinite, it must intersect $\cap_{e\in F^c} A_e$. And changing the state of any edge within or near F can only affect the cluster within or near F. It cannot affect whether an infinite cluster exists entirely in F^c. Thus E remains unchanged if the states within F are altered, so C is independent of $\sigma(A_e : e \in F)$. Since the choice of F is arbitrary, and configuration of any finite F does not affect the occurence of E, we take $F_k := \{e_1, \ldots, e_k\}$. This gives that, $E \in \tau$. By Kolmogorov's $0-1$ law, $\mathbb{P}(E) = 0$ or 1. Since $p > p_c(G_f)$, we get $\Theta_{G_f}(p) > 0$. Also $\{\#C_{G_f}(0) = \infty\} \subseteq E$. So we must have $\mathbb{P}(E) = 1$. □