Percolation Theory

Homework 9

BIKRAM HALDER
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Let T be a measure preserving transformation on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that a set $A \in \mathcal{F}$ is *T-invariant* if $T^{-1}(A) = A$. We say that a set $A \in \mathcal{F}$ is *almost surely T-invariant* if $\mathbb{P}(A \Delta T^{-1}(A)) = 0$.

Problem 1. Show that $\mathcal{I} = \left\{A \in \mathcal{F} \colon T^{-1}(A) = A\right\}$ is a σ-algebra. (This is called the *invariant σ-algebra* of T.)

Solution. Obviously $T^{-1}(\Omega) = \Omega$ and $T^{-1}(\emptyset) = \emptyset$. For $A \in \mathcal{I}$, $T^{-1}(A^c) = \left(T^{-1}(A)\right)^c = A^c$. Let $\{A_n\}_{n\geq 1}$ be a sequence of T-invariant sets. Then $T^{-1}\left(\bigcup_{n\geq 1} A_n\right) = \bigcup_{n\geq 1} T^{-1}(A_n) = \bigcup_{n\geq 1} A_n$. So $\bigcup_{n\geq 1} A_n$ is T-invariant. Thus $\mathcal{I}$ is a σ-algebra. □

Problem 2. Show that the collection of almost surely T-invariant sets is just a completion of $\mathcal{I}$.

Solution. Let $\mathcal{I}_0 = \left\{A \in \mathcal{F} \colon \mathbb{P}(A \Delta T^{-1}(A)) = 0\right\}$. Note that $\mathcal{I} \subseteq \mathcal{I}_0$. Let $A \in \mathcal{I}_0$. Then

$$\mathbb{P}(A^c \Delta T^{-1}(A^c)) = \mathbb{P}\left(A^c \Delta \left(T^{-1}(A)\right)^c\right) = \mathbb{P}(A \Delta T^{-1}(A)) = 0.$$

So $A^c \in \mathcal{I}_0$. Let $\{A_n\}_{n\geq 1}$ be a sequence of almost surely T-invariant sets. Then

$$\begin{aligned}
\mathbb{P}\left(\left(\bigcup_{n\geq 1} A_n\right) \Delta T^{-1}\left(\bigcup_{n\geq 1} A_n\right)\right) &= \mathbb{P}\left(\left(\bigcup_{n\geq 1} A_n\right) \Delta \bigcup_{n\geq 1} T^{-1}(A_n)\right) \\
&\leq \sum_{n\geq 1} \mathbb{P}\big(A_n \Delta T^{-1}(A_n)\big) \\
&= 0.
\end{aligned}$$

So $\bigcup_{n\geq 1} A_n$ is almost surely T-invariant. Thus $\mathcal{I}_0$ is a σ-algebra. Now let $A \in \mathcal{I}_0$ such that $\mathbb{P}(A) = 0$ and $B \subseteq A$. Then

$$\mathbb{P}(B \Delta T^{-1}(B)) \leq \mathbb{P}(A \Delta T^{-1}(A)) = 0.$$

So $B \in \mathcal{I}_0$. Thus $\mathcal{I}_0$ is a completion of $\mathcal{I}$. □