**INDIAN STATISTICAL INSTITUTE, KOLKATA**
**Assignment 1 , Second Semeter 2022-23**
**Algebra , M. Math I**
**Date :**

1. Let $H \subset G$ be a subgroup. We define a relation on $G$ as follows : $g_1 \sim g_2$ if $g_2^{-1} g_1 \in H$. This is an equivalence relation on $G$.
   (a) This is an equivalence relation on $G$.
   (b) $g_1 \sim g_2$ iff $g_1 \in g_2.H$ iff $g_2 \in g_1 H$ iff $g_1 H = g_2 H$.
   (c) For $g, g' \in g_1 H$, we have $g \sim g'$.
   (d) There is a natural bijection $H \to g.H$ given by $h \mapsto g.h$.
   (e) The equivalence class of $g$ under the relation $\sim$ is precisely the set $g.H$.
2. Find out all the subgroups of $S_3$. Which subgroups are normal ?
3. Give an example of a group of infinite cardinality, such that each element of the group has finite order.
4. List all the subgroups of $\mathbb{Z}/2023\mathbb{Z}$.
5. Let $f(x) = x^3 + px + q$, and let $\alpha_1, \alpha_2, \alpha_3$ be three roots of $f(x)$, then
$$\triangle := -(4p^3 + 27q^2) = ((\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_1))^2.$$
6. Let $f \in \mathbb{Q}[t]$ and $f \neq 0$. Let $\alpha_1, \ldots, \alpha_k$ be all the distinct roots of $f$ in $\mathbb{C}$.
$$Gal(f) := \left\{ \sigma \in S_k | (\alpha_1, \ldots, \alpha_k) \text{ and } (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)}) \text{ are } \mathbb{Q} \text{ conjugates} \right\}.$$
   Show that $Gal(f) \subset S_k$ is a subgroup.
7. Construct an example of $f(x) \in \mathbb{Q}[x]$ of degree 3 with
   (a) $Gal(f) = \{e\}$.
   (b) $Gal(f) = \mathbb{Z}/2\mathbb{Z}$.
   (c) $Gal(f) = \mathbb{Z}/3\mathbb{Z}$.
   (d) $Gal(f) = S_3$.
8. Let $G$ and $H$ be finite groups such that $gcd(|G|, |H|) = 1$. Show that any group homomorphism $\phi : G \to H$ (resp. $\psi : H \to G$) is trivial.
9. Let $G$ be any group and let $g \in G$ such that $ord(g) < \infty$. Show that for any $m < ord(g)$ we have $ord(g^m) = ord(g)/gcd(m, ord(g))$.

10. Show that $(\mathbb{Z}[x], +)$ and $(\mathbb{Q}^+, .)$ ( positive rationals with multiplication as binary operation ) are isomorphic.

11. Determine whether the following pair of groups are isomorphic.

    (a) $(\mathbb{R}^*, .)$ and $(\mathbb{C}^*, .)$.

    (b) $\mathbb{Z}/6\mathbb{Z}$ and $S_3$.

12. Let $G$ be a group and $X$ be a set. Show that there is a one to one correspondence between $G$ action on $X$ and group homomorpshims $\phi : G \to S_X$.

13. Let $X$ be a topological space and let $x \in X$ be a point. Let

$$\Omega_x(X) := \{f : [0,1] \to X | f(0) = f(1) = x, f \text{ continuous}\}.$$

Let $f, g \in \Omega_x(X)$. Then define

$$g \circ f(t) := \begin{cases} f(2t), & 0 \leq t \leq 1/2 \\ g(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

Let $e : [0,1] \to X$ be the constant loop, that is $e(t) = x, t \in [0,1]$ and let for any $f \in \Omega_x(X)$ we define $f^{-1}(t) := f(1-t)$. Show the following

(a)

$$h \circ (g \circ f)(t) := \begin{cases} f(4t), & 0 \leq t \leq 1/4 \\ g(4t-1), & 1/4 \leq t \leq 1/2 \\ h(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

(b)

$$(h \circ g) \circ f(t) := \begin{cases} f(2t), & 0 \leq t \leq 1/2 \\ g(4t-2), & 1/2 \leq t \leq 3/4 \\ h(4t-3), & 3/4 \leq t \leq 1 \end{cases}$$

(c) Show that there exists a continous map $F : [0,1] \times [0,1] \to X$, Such that $F(0,t) = F(1,t) = x$ and $F(t',0) = h \circ (g \circ f)(t')$ and $F(t',1) = (h \circ g) \circ f(t')$.

(d) Show that $f \circ e \neq f$ in general similarly $e \circ f \neq f$ in general.

(e) Show that in all of these cases we have equality upto base point preserving homotopy.