INDIAN STATISTICAL INSTITUTE, KOLKATA
Assignment 10 , Second Semester 2024-25
Algebra , M. Math I
Date :

1. Show that trisecting an angle is not possible using straightedge and compass.
2. Show that the unit circle can not be squared using straightedge and compass.
3. Let p be a prime. Show that a regular p-gon of unit side length can be constructed using straightedge and compass if and only if $p = 2^{2^n} + 1$ for some non negative integer n.
4. Is it possible to construct a square using straightedge and compass, whose area is equal to the area of a given triangle?
5. Is it possible to construct a cube whose volume is equal to the volume of a regular tetrahedron whose sides are equal to 1?
6. Show that for any finite extension K/k, the cardinality of $Aut(K/k)$ divides the degree $[K : k]$.
7. Let K/k be an algebraic extension. Let K_{sep} be the set of elements over k which are seprable over k.
    (a) Show that K_{sep} is a subfield of K containing k. It is called the seprable closure of k i n K.
    (b) Show that an algebraic extension K/k is purely inseparable if and only if $K_{sep} = k$.
    (c) Show that for any algebraic extension K/k, the extension K/K_{sep} is purely inseprable.
8. Describe the intermediate fields of $\mathbb{Q}(\zeta_7)/\mathbb{Q}$, where $\zeta_7 = e^{2\pi i/7}$. That is show the following
    (a) Show that $\mathbb{Q}(\zeta_7)/\mathbb{Q}$ is Galois and the Galois group is isomorphic to $(\mathbb{Z}/7\mathbb{Z})^*$. Infact , a genrator of $G(\mathbb{Q}(\zeta_7)/\mathbb{Q})$ can be chosen to be the $\sigma \in G(\mathbb{Q}(\zeta_7)/\mathbb{Q})$ such that $\sigma(\zeta_7) = \zeta_7^3$.
    (b) The two non trivial subgroups of $G(\mathbb{Q}(\zeta_7)/\mathbb{Q})$ are (σ^3) and (σ^2). Compute $\sigma^3(\zeta_7^j)$ and $\sigma^2(\zeta_7^j)$.
    (c) The element $\zeta_7 + \overline{\zeta}_7$ is fixed by σ^3. Conclude that $\mathbb{Q}(\zeta_7 + \overline{\zeta}_7)$ is the fixed field of (σ^3).
    (d) Show that $\zeta_7 + \zeta_7^2 + \zeta_7^4$ is fixed by σ^2. Conclude that the fixed field of (σ^2) is $\mathbb{Q}(\sqrt{-7})$.

9. Let K_1/k be a Galois extension and K_2/k be another extension, such that K_1, K_2 both are subfield of a common extension L/k.

   (a) Show that $K_1/K_1 \cap K_2$ is Galois and the homomorphism $G(K_1K_2/K_2) \to G(K_1/K_1 \cap K_2)$, given by $\sigma \mapsto \sigma|_{K_1}$ is an isomorphism. (For the surjectivity use $G(K_1/K_1^{G'}) = G'$ for any subgroup $G' \subset G(K_1/K_1 \cap K_2)$.)

   (b) Show that $[K_1K_2 : k] = \frac{[K_1:k][K_2:k]}{[K_1 \cap K_2:k]}$.

   (c) Show that if $gcd([K_1 : k], [K_2 : k]) = 1$, then $[K_1K_2 : k] = [K_1 : k][K_2 : k]$.

10. Describe the Galois group of the Splitting field of $x^5 - 2$ over $\mathbb{Q}$. That is , show the following

   (a) Show that, the splitting field is $\mathbb{Q}(\zeta_5, 2^{1/5})$, where $\zeta_5 = e^{2\pi i/5}$ and $2^{1/5}$ is real 5-th root of 2. Show that $[\mathbb{Q}(\zeta_5, 2^{1/5}) : \mathbb{Q}] = 20$.

   (b) Show that the subgroup N which fixes the sub field $\mathbb{Q}(\zeta_5)$ is normal of order 5.

   (c) Let $G := G(\mathbb{Q}(\zeta_5, 2^{1/5})/\mathbb{Q})$ and let H be the subgroup of G whose fixed field is $\mathbb{Q}(2^{1/5})$. Then show that $G/N \cong (\mathbb{Z}/5\mathbb{Z})^* \cong H$. A generators $\tau \in H$ such that $\tau(\zeta_5) = \zeta_5^2$ and let σ be a generator of N such that $\sigma(2^{1/5}) = \zeta_5 2^{1/5}$.

   (d) Show that $\sigma^5 = 1, \tau^4 = 1, \tau\sigma\tau^{-1} = \sigma^2$.

   (e) How many conjugates does H have in G ? Find the fixed fields of the conugates of H.