INDIAN STATISTICAL INSTITUTE, KOLKATA
Assignment 11 , Second Semeter 2024-25
Algebra 2 , M1
Date :

1. Let G be a finite group and let d be divisor of $|G|$. Suppose there are exactly two subrgoups H_1, H_2 of G of cardinality d, such that H_1 is not normal.
   (a) Show that $|G|$ is even.
   (b) Show that $H_1 \cong H_2$ and $N_G(H_1) \cong N_G(H_2)$.
   (c) Show that $|G| = |N_G(H_1)| + |N_G(H_2)|$.
2. Let G be a finite group. Show that there exists a a filed K and finite Galois extension L/K such that $G(L/K) \cong G$.
3. Let p be a prime show that there exists $K/\mathbb{Q}$ finite and K'/K Galois such that $G(K'/K) \cong \mathbb{Z}/p\mathbb{Z}$ and $[K : \mathbb{Q}] = p!/2$.
4. Construct a Galois extension $K/\mathbb{Q}$ of degree 5.
5. Let p be a prime and $a \in \mathbb{Z}$ positive such that $p|a$ but p^2 doe snot divide a.
   (a) Show that $x^m - a$ is an irreducible polynomial in $\mathbb{Q}[x]$.
   (b) Let ζ_m be a primitive m-th root. Show that $[\mathbb{Q}(\zeta_m) : \mathbb{Q}] = \phi(m)$ and $x^m - a$ does not have a root in $\mathbb{Q}(\zeta_m)$.
   (c) Show that $\mathbb{Q}(\zeta_m, a^{1/m})/\mathbb{Q}(\zeta_m)$ is a cyclic Galois extension.
   (d) Find an irreducible polynomial $f(x) \in \mathbb{Q}(\zeta_m)[x]$, such that $\mathbb{Q}(\zeta_m, a^{1/m})$ is the splitting field of $f(x)$ over $\mathbb{Q}(\zeta_m)$.
   (e) Let m be a prime and let $g(x) \in \mathbb{Q}[x]$ be the minimal polynomial of ζ_m over $\mathbb{Q}$. Show that $g(x)$ is irreducible over $\mathbb{Q}(a^{1/m})$ and $x^m - a$ is irreducible over $\mathbb{Q}(\zeta_m)$.
6. Let p be a prime, n positive integer and let k be finite field with $q = p^n$ elements. Let K/k be a extension of degree 2. Let $G(K/k) = (\sigma)$, where $\sigma(x) = x^q$.
   (a) Show that the map $\phi : K^* \to K^*$, given by $\phi(x) := \sigma(x)/x$, is a group homomorphism.
   (b) Show that $N_{K/k} : K^* \to k^*$ is a group homomorphism.
   (c) Use Hilbert theorem 90 to show that $|Ker(N_{K/k})| = q + 1$.