**INDIAN STATISTICAL INSTITUTE, KOLKATA**
**Assignment 2 , Second Semester 2022-23**
**Algebra , M. Math I**
**Date :**

1. Show that an isometry $h : \mathbb{R}^n \to \mathbb{R}^n$ can be uniquely written as $h = t_u \circ k$ for some $u \in \mathbb{R}^n$ and $k$ is a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ such that the matrix of $k$ , say $A$, with respect to the standard basis satisfies $A^T A = Id$. Using this or otherwise, show that the set of isometries $Isom(\mathbb{R}^2)$ is a group with composition of isometries being the group operations.

2. Show that $Isom(\mathbb{R}^2)$ is isomorphic to a subgroup $H$ of $Gl_3(\mathbb{R})$ such that the natural action of $Isom(\mathbb{R}^2)$ on $\mathbb{R}^2$ corresponds to, via the isomorphism, an action of the subgroup $H$ on the subset $\{(v, 1) \in \mathbb{R}^3 | v \in \mathbb{R}^2\}$. (hint :
$$H := \left\{ \begin{bmatrix} A & w \\ 0 & 1 \end{bmatrix} |A \in Gl_2(\mathbb{R}), A.A^t = Id, w \in \mathbb{R}^2 \right\}$$
)

3. Let $f \in Isom(\mathbb{R}^2)$ such that $f(v) = Av + w$, with $A \in Gl_2(\mathbb{R})$ with $A.A^t = Id$ and $w \in \mathbb{R}^2$. Take the natural action of $Isom(\mathbb{R}^2)$ on $X = \mathbb{R}^2$. Describe $X^f$ in each of the following cases :
   (a) (Identity) $A = Id$, $w = 0$.
   (b) (non zero translation) $A = Id$, $w \neq 0$.
   (c) (nonzero rotation) $det(A) = 1$ and $A \neq Id$.
   (d) (reflection) $det(A) = -1, Aw = -w$.
   (e) (glide reflection) $det(A) = -1, Aw \neq -w$.

4. Consider the terminology of the previous problem. Show that composition of two reflections is a translation or a rotation. Show that every rotation is a composition of two reflections.

5. Show that the glide reflections are composition of 3 and no fewer reflections. ( Hint : Show that every translation is a composition of two reflections. Then use the previous problem).

6. A group $G$ is said to be acting freely on a said $X$ if for every $x \in X$, the equation $g.x = x$ implies $g = e$. For an action of a group $G$ on $X$, show that the following are equivalent
   (a) $G$ acts freely on $X$.
   (b) For any two elemenst $x, y \in X$ there is at most one $g \in G$ such that $g.x = y$.

(c) The map $G \times X \to X \times X$ defined as $(g, x) \mapsto (x, g.x)$ is injective.

7. An action of a group $G$ on a set $X$ is called transitive if or any two elements $x, y \in X$, there exists atleast one $g \in G$ such that $gx = y$.Show that the following are equivalent for an action of a group $G$ on $X$.

   (a) The action of $G$ on $X$ is transitive.

   (b) $|X/G| = 1$.

   (c) The map $G \times X \to X \times X$ defined as $(g, x) \mapsto (x, g.x)$ is surjective.

   Conclude that for a free and transitive action of $G$ on $X$ and $x \in X$, there exists a group structure on $X$ induced by the action of $G$ on $X$ such that

   (a) $x$ is the identity element and

   (b) there exists a group isomorphism isomorphism $\phi : G \to X$ such that $\phi(e_g) = x$.

8. Show that a group of order $p^2$ is abelian. Give an example of a group of order $p^3$ which is not abelian.

9. Let $G$ be a finite group such that $p||G|$ for a prime $p$. Show by induction on the cardinality of $G$ that there exists an elemnt of order $p$.

10. Let $G$ be a finite group and $p$ be a prime

$$X := \{(x_1, \dots, x_p) \in G^p | x_1.x_2 \dots x_p = e\} .$$

   (a) Show that the cyclic group $C_p$ of order $p$ acts on $X$ as cyclic permutation.

   (b) Give examples of orbits of each possible size.

   (c) Show that there exists atleast two trivial orbits, i.e. two orbits of cardinality 1 whenever $p||G|$.

   (d) Conclude that whenever $p||G|$, there exists atleast $p - 1$ element of order $p$.

11. Let $p : E \to X$ be a covering space of a topological space $X$. Let $x \in X$ and let

$$S := p^{-1}(x) := \{y \in E | p(y) = x\} .$$

   Let $G = \pi_1(X, x)$.

   (a) Using unique path lifting and unique homotopy lifting property construct an action of $G$ on $S$.

   (b) Let $y \in S$, describe with proof $Stab(y)$.

   (c) Let $E$ be connected. Show that the action of $G$ on $S$ is transitive.

(d) Let $E$ be connected. Show that the action of $G$ on $S$ is free iff $\pi_1(E, y) = \{e\}$ for every $y \in S$.

12. Let

$$Gr(k, n) := \{V \subset \mathbb{R}^n | \text{subspace }, dim(V) = k\}.$$

(a) Let $G = GL_n(\mathbb{R})$. Show that the map given by $(T, V) \mapsto T(V)$ where $T \in GL_n(\mathbb{R})$ and $V \in Gr(k, n)$ gives an action $G$ on $Gr(k, n)$.

(b) Show that the action is transitive.

(c) For $k = 1$, and $l \in Gr(1, n)$, show that

$$Stab(l) = \{T \in GL_n(\mathbb{R}) | T(v) = \lambda.v, |v| = 1, \lambda \in \mathbb{R}, v \in l\}.$$

13. Let $|G| = p^n.m$, $p$ is a prime, $(p, m) = 1$. Show that for any $k \leq n$, there exists a subgroup $H$ of $G$ such that $|H| = p^k$. (Hint : Use the first part of the Sylows theorem , may be you can use some technique from the proof).