INDIAN STATISTICAL INSTITUTE, KOLKATA
Assignment 3 , Second Semeter 2024-25
Algebra 2 , M1
Date :

1. Let $X := \{1, 2, 3, 4, 5\}$ and $G \subset S_5$ given by $G := \{e, (123), (132), (45), (123)(45), (132)(45)\}$. Find out the distinct orbits. Find out G_x for $x \in X$ and X^g for $g \in G$.

2. Let G_1, G_2 be groups. Let $Hom(G_1, G_2) := \{\phi : G_1 \to G_2 | \phi \ homomorphism\}$. Gind out the cardinality of $Hom(G_1, G_2)$ in the following cases
   (a) $G_1 = \mathbb{Z}/5\mathbb{Z}$ and $G_2 = \mathbb{Z}/17\mathbb{Z}$.
   (b) $G_1 = \mathbb{Z}/p\mathbb{Z}$ and $G_2 = \mathbb{Z}/q\mathbb{Z}$ where p and q not necessarily distinct primes.
   (c) $G_1 = \mathbb{Z}/p\mathbb{Z}$ and $G_2 = \mathbb{Z}/(p.q)\mathbb{Z}$ where p and q are distinct primes.
   (d) $G_1 = \mathbb{Z}/(p.q)\mathbb{Z}$ and $G_2 = \mathbb{Z}/p\mathbb{Z}$ for p and q distinct primes.
   (e) G_1 and G_2 such that $gcd(|G_1|, |G_2|) = 1$.

3. How many $\mathbb{Z}/5\mathbb{Z}$ actions are there on the set $X := \{1, 2, 3, 4, 5, 6, 7\}$.

4. Let $X := \{1, 2, \ldots, n\}$ and take the canonical action of $a : S_n \times X \to X$ given by $a(\sigma, i) := \sigma(i)$. Let $x \in X$ be an element. Find out the carfinality of $(S_n)_x := \{\sigma \in S_n | a(\sigma, x) = x\}$.

5. Let $H, N \subset G$, be subgroups. Show that $H.N$ is subgroups iff $H.N = N.H$.

6. Let $\pi : G \to G'$ be a group homomorphism and let $H' \subset G'$ be a subgroup of G'. Show that $\pi^{-1}(H')$ is subgroup of G. Show that if H' is normal then $\pi^{-1}(H')$ is normal.

7. (third isomorphism theorem) Let G be a group and $N, H \subset G$ normal subgroups of G such that $N \subset H$, then $G/H \cong G/N/(H/N)$.

8. Let $\phi : G \to G'$ be a group homomorphism. Let $H \subset G$ be a subgroup. Show that $\phi(H) \cong H/(H \cap ker(\phi))$.

9. Show that
   (a) (rotation by angle θ) The orthogonal 2×2 matrices with determinant 1 are the following
   $$\begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$
   (b) (reflection with axis the line with angle $1/2\theta$ with the x axis.) The orthogonal 2×2 matrices with determininat -1 are the matrices
   $$\begin{bmatrix} cos(\theta) & sin(\theta) \\ sin(\theta) & -cos(\theta) \end{bmatrix}$$

10. From the definition of D_n, deduce that D_n is a group.

11. Let $r \in D_n$ be the counterclockwise rotation by angle $2\pi/n$ and let s be any reflection. Show that $srs = r^{-1}$. Show that, the n reflections are given by $s, rs = sr^{-1}, \ldots, r^{n-1}s = sr^{-n+1}$.

12. Find out the center of D_n.

13. Let $X_{n,e}$ be the set of sides of R_n. Construct a non trivial action of D_n on $X_{n,e}$. Given any $x \in X_{n,e}$ find out O_x and $Stab(x)$. Let $g \in D_n$, find out $X^g_{n,e}$.

14. Let $X_{n,v}$ be the set of vertices of R_n. Construct a non trivial action of D_n on $X_{n,v}$. Given any $x \in X_{n,v}$ find out O_x and $Stab(x)$. Let $g \in D_n$, find out $X^g_{n,v}$.