INDIAN STATISTICAL INSTITUTE
Assignment 4
Algebra 2, M1

1. Let G be a non trivial finite abelian group

   (a) Show that for any element $g \in G$ and any positive integer $d|o(g)$, we have $o(g^d) = o(g)/d$.

   (b) Define
   $$m := max(\{o(g)|g \in G\}).$$
   Show that for any $g \in G$, $g^m = e$.

2. Let G be a finite group.

   (a) Show that if $G/Z(G)$ is cyclic then G is abelian.

   (b) Let G be abelian, p be a prime such that $|G| = p^k$ and $k \geq 1$, then there exists a $m = p^l$ such that, $k \geq l \geq 1$, $g^m = e$ for all $g \in G$, and there exists an element $h \in G$ such that $o(h) = m$.

   (c) Let G be abelian, p be a prime such that $|G| = p^k, k \geq 1$ and let m be as in the last part. Then show that G is cyclic iff $|G| = m$.

   (d) Let G be abelian, p be a prime such that $|G| = p^k$, let m be as in the last part, let $h \in G$ such that $o(h) = m$, and let $(h) \neq G$ (here (h) is the cyclic subgroup generated by h). Show that there exists $x \in G, x \notin (h)$ such that $x^p = h^j$ for some j. (Hint : Apply Cauchy's thorem on $G/(h)$).

   (e) Use the previous parts to show that if G is an abelian group, $|G| = p^k$, then G is cyclic iff there exists a unique subgroup of G of cardinality p. (Hint : the unique subgroup of order p will be contained inside the (h) of the previous problem and use part (b)).

   (f) Use induction and previous parts to show that for any group G (not necessarily abelian) of cardinality p^k, G is cyclic iff for any p^l with $l \leq k$, there exists a unique subgroup $H_l \subset G$ of cardinality p^l.

   (g) Let G be a group such that $|G| = p^a.q^b$, where p and q are distinct primes. Show that G is cyclic iff for any $d|p^a.q^b$, there exists a unique subgroup $H_d \subset G$ of cardinality d. (Hint: Use Sylow theorems.)