INDIAN STATISTICAL INSTITUTE, KOLKATA
Assignment 5 , Second Semester 2024-25
Algebra , M. Math I
Date :

1. Let G be a group. Show that if $|G| < \infty$ then G is a group of order p^n for a prime p (i.e. it is a p-group) iff every non trivial element of g has order p^k for $k > 0$.

2. Show that disjoint cycles in S_n commute with each other. Let $\tau_m, \sigma_k \in S_n$ be disjoint cycles of length m and k respectively. Then compute the order of $\tau_m \circ \sigma_k$.

3. Let $G = \mathbb{Z}/n$ and let $g \in G$ be an element of order m. Find out the cycle decomposition of the element $\sigma_g \in S_n = Bij(\mathbb{Z}/n)$, given by $\sigma_g(i) = g.i(mod)n$.

4. Show that every element $\sigma \in S_n$ can be writtes as composition of transposition. For a fixed $\sigma \in S_n$ let n_σ, n'_σ be number of transpositions appearing in two such composition for σ. Then show that $n_\sigma = n'_\sigma(mod)2$.

5. Show that for any two k-cycles $(i_1 i_2 \cdots_k)$ and $j_1 j_2 \dots j_k)$ and $k \leq n$ we have a $\sigma \in S_n$ such that
$$\sigma(i_1 i_2 \dots i_k)\sigma^{-1} = (\sigma(i_1) \dots \sigma(i_k)) = (j_1 j_2 \dots j_k).$$
Conclude that all cycles of same length in S_n are conjugate. Show that (123) and (132) are not conjugate in A_n.

6. Show that two permutations in S_n are conjugate iff they have the same cycle type.

7. Let G be a group and let $g \in G$ and let h be an element conjugate to g. Show that $ord(g) = ord(h)$. Given example of group G, elements $g, h \in G$ such that $ord(g) = ord(h)$ but g is not conjugate to h.

8. Write down the class equation of D_n. More explicitly, describe the distinct conjugacy classes of D_n.

9. Write down the class equations of S_3, A_3, S_4, A_4.

10. Let G, H, K be groups such that we have the following exact sequence of groups
$$1 \to H \to^i G \to^p K \to 1,$$
and let $s : K \to G$ be a group homomorphism such that $p \circ s = id_K$. Show that
$$G \cong i(H) \rtimes_\phi s(K),$$
where $\phi : s(K) \to Aut(i(H))$ is the conjugation action. Show that if ϕ is trivial thenthere exists $r : g \to H$ group homomorphism such that $r \circ i = id_H$.