INDIAN STATISTICAL INSTITUTE, KOLKATA
Assignment 6 , Second Semeter 2024-25
Algebra 2 , M1
Date :

1. Let F be a field. Show that there are infinitely many irreducible polynomial in $F[x]$.

2. Show that $x^p - x$ has p distinct roots in $\mathbb{Z}/p\mathbb{Z}$.

3. Let k be a field and p be a prime such that $char(k) = p$. Show that all the roots $x^p - a$ are equal in any splitting field of $x^p - a$ over k. Conclude that if $x^p - a$ does not have any root in k then it is irreducible over k.

4. Let $k = \mathbb{F}_p$.
   (a) Let L/k be a splitting field of $f(x) = x^p - x + a$, where $a \neq 0$. Show that all the roots of $f(x)$ are distinct in L.
   (b) Show that $f(x) = f(x+1)$.
   (c) Show that if $\alpha \in L$ is a root of f then $\alpha + j$ is also a root of $f(x)$ for $j \in \mathbb{F}_p$.Conclude that f does not have any root in $\mathbb{F}_p$.
   (d) Let g be any irreducible factor of f in $\mathbb{F}_p[x]$ and $\alpha \in L$ be any root of f in L. Then show that the sum of roots of g is $k\alpha + r$, where $k = deg(g)$ and $r \in \mathbb{F}_p$, $k\alpha \neq 0$. Conclude that f is irreducible.

5. Let p be a prime and let $f(x) = \sum_{i=0}^{p-1} x^i$. Find out a splitting field K of $f(x)$ over $\mathbb{Q}$. Show that $[K : \mathbb{Q}] = p - 1$.

6. Let $f(x) = x^3 - 2$. Find out a splitting field K of f over $\mathbb{Q}$. Show that $[K : \mathbb{Q}] = 6$.

7. Construct a splitting field K of $x^4 - 2$ over $\mathbb{Q}$. Find out $[K : \mathbb{Q}]$.

8. Show the following :
   (a) A field of cadinality p^k contains a subfeild of order p^r iff $k|r$.
   (b) Irreducible factors of the polynomial $x^q - x$ over $\mathbb{F}_p$ are the irreducible polynomials $\mathbb{F}_p[x]$ whose degrees divide r, here $q = p^r$.

9. Let $f(x) \in \mathbb{F}_p[x]$ be an irreducible polynomial of degree n, and let $q = p^n$. Show that $f(x)|x^q - x$. Conclude that $f(x)$ can not have repeated roots in any splitting field of $f(x)$ over $\mathbb{F}_p$.