INDIAN STATISTICAL INSTITUTE, KOLKATA
Assignment 7 , Second Semester 2024-25
Algebra , M. Math I
Date :

1. Let K/k be an algebraic extension and let $k \subset A \subset K$ be a subring of K containing k. Show that A is a field. Construct an example of a transcendental extension K/k and subring $A \subset K$ containing k, such that A is not a field.

2. Show that if $[k(\alpha) : k]$ is odd, then $k(\alpha) = k(\alpha^2)$.

3. Show that any field extension K/k with $[K : k] = 2$ is normal.

4. Construct a splitting field of $x^5 - 2$ over $\mathbb{Q}$. What is its degree over $\mathbb{Q}$.

5. Show that an algebraic extension N/k is normal if and only if for any $\alpha \in N$, the number of k-homomorphism $\phi : k(\alpha) \to N$ is equal to the number of distinct roots of the minimal polynomial of α.

6. An irreducible polynomial $f \in k[x]$ is called normal if a splitting field N of f over k is obtained by adjoining only one zero of f, that is, $N = k(\alpha)$ where $f(\alpha) = 0$. Show that an irreducible polynomial $f \in k[x]$ is normal if and only if its splitting field over k has degree equal to the degree of f.

7. Let $k \subset K \subset L$ be finite exetnsions such that L/k is normal, show that L/K is normal.

8. Which of the following extension L/k are normal ?

   (a) $L = \mathbb{F}_3(t)$ and $k = \mathbb{F}_3(t^2)$.

   (b) $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $k = \mathbb{Q}$.

9. For the following extension L/k find the smallest normal extension N/k such containg L.

   (a) $L = \mathbb{Q}(2^{1/4})$ and $k = \mathbb{Q}$.

   (b) $L = \mathbb{Q}(t)$ and $k = \mathbb{Q}(t^4)$.

10. Let k be a field of characteristic p and let $f(x) \in k[x]$ be irreducible. Show that $f(x)$ can be written as $f(x) = g(x^{p^l})$ with $g(x)$ irreducible and separable in $k[x]$. Moreover, deduce that every root of $f(x)$ has the same multiplicity p^l in any splitting field.