INDIAN STATISTICAL INSTITUTE, KOLKATA
Assignment 9 , Second Semester 2024-25
Algebra , M. Math I
Date :

1. Let K/k be a finite field extension let $K \subset N$ be an extension such that N/k is finite normal. Show that there exists a subfield $N_K \subset N$ containing K, which is the smallest normal extension N_K/k containing K. This N_K will be called the normal closure of K in N over k. Show that N_K is the intersection of all the normal extension N'/k containing K and contained in N.

2. Let K/k be a finite seprable extension. Let L/k be subfield of K. Show that K/L is separable, and L/k is separable.

3. Let K be a finite field with p^n elements. Show that every element of K has a unique p-th root in K.

4. Let K/k be a finite extension and let N/K be a finite extension such that N/k is normal. Define
$$[K : k]_{s,N} := |Hom_k(K, N)|.$$
   (a) Show that for any other N'/K finite extension such that N'/k is normal we have $[K : k]_{s,N'} = [K : k]_{s,N}$. So from now on we will denote $[K : k]_s$ to be that number.
   (b) Show that for $k \subset K_1 \subset K$ finite extensions, we have
$$[K : k]_s = [K : K_1]_s[K_1 : k]_s.$$
   (c) Show that K/k finite extension is separable iff $[K : k]_s = [K : k]$.
   (d) Show that $[K : k]_s | [K : k]$ for any finite extension K/k. Show that the quotient is 1 that is $[K : k]/[K : k]_s = 1$ if $char(k) = 0$ else it is a power of some prime p for $p > 0$.
   (e) Let K/k be nontrivial finite extension. Show that $[K : k]_s = 1$ iff for every $\alpha \in K \setminus k$ is inseperable.
   (f) Show that $[k(\alpha) : k]_s = 1$ then the minimal polynomial of f has only one root.
   (g) Let $f \in k[x]$ be an inseparable irreducible polynomial. Let α be a root of f in a splitting field of f over k. Let $[k(\alpha) : k]_s = 1$, then $f(x) = x^{p^n} - a$ for some $n > 0$ and p-prime.
   (h) Let K/k finite normal extension and let $G(K/k)$ be the group of k-automorphism of K. Show that
$$[K^G : k]_s = 1, [K : K^G]_s = [K : K^G].$$