1. Questions for practice

Notation: Ω is a connected open subset of $\mathbb{C}$. $f \in H(\Omega)$ means f is holomorphic on Ω. ∂S is the boundary of a subset S of $\mathbb{C}$. $\mathbb{D}$ is unit disc, $D(a, r)$ is disc of radius r, centered at a, $\mathbb{H}$ is the upper half space.

Terminology: A bi-holomorphic function $f : \Omega_1 \to \Omega_2$ is a bijective holomorphic function, (consequently) f^{-1} is also holomorphic. A conformal map $f : \Omega_1 \to \Omega_2$ is a holomorphic function such that $f'(z) \neq 0$ for all $z \in \Omega_1$, thus it is only locally $1 - 1$. What is an example of a conformal map which is not bi-holomorphic?

(1) Establish a Schwartz lemma for these cases: [If you find it difficult, try only the first part of the Schwartz lemma.]
   - (a) $f : \mathbb{H} \to \mathbb{D}$ a holomorphic function such that $f(i) = 0$,
   - (b) $f : \mathbb{H} \to \mathbb{H}$ a holomorphic function such that $f(i) = i$.
   - (c) $f : \mathbb{D} \to \mathbb{D}$ a holomorphic function such that $f(a) = 0$ for some $a \in \mathbb{D}$.
   - (d) $f : \mathbb{D} \to \mathbb{D}$ a holomorphic function such that $f(0) = a$ for some $a \in \mathbb{D}$.
   - (e) $f : \mathbb{D} \to \mathbb{D}$ a holomorphic function such that $f(a) = a$ for some $a \in \mathbb{D}$.
   - (f) $f : \mathbb{D} \to \mathbb{D}$ a holomorphic function such that $f(a) = b$ for some $a, b \in \mathbb{D}$.
   - (g) $f : D(0, r) \to D(0, R)$ a holomorphic function such that $f(0) = a$ for some $a \in D(0, R)$.

(2) Establish an argument principle when the closed curve γ is not a simple closed curve.

(3) Establish a Rouchés theorem when the function has zeros and poles inside the closed curve. What happens if the closed curve is not simple?

(4) Establish a Johnson's formula when the function f has zeros and poles inside $\mathbb{D}$. (Add necessary hypothesis.) [As a toy case, assume that f has only two simple poles and two simple zeros in $\mathbb{D}$. Use ψ_α functions.]

(5) Let C be a circle parametrized by $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$ in a region Ω which contains C and its interior. Take $f \in H(\Omega)$ such that f has n zeros (counting multiplicities) inside C and f does not vanish on C. Consider the closed curve $f(\gamma(t)) = \Gamma(t)$ on $\mathbb{C}$. Show that the winding number of Γ around 0, $W_\Gamma(0) = n$.

(6) $F \in H(\Omega)$ and $\Omega \supset \bar{\mathbb{D}}$. Let f be the restriction of F on the unit circle $\partial\mathbb{D}$. Considering f as a function on the circle show that $\widehat{f}(n) = 0$ if $n < 0$ where

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta.$$

(7) Verify that $f : z \mapsto e^{iz}$ is a conformal map from $\mathbb{H} \to \mathbb{D}$ (see definition above).

(8) Let $S = \{z \in \mathbb{C} \mid 0 < \Im z < \pi\}$. Show that $F : z \mapsto \log z$ is a conformal map from $\mathbb{H} \to S$. Find a conformal map from $\mathbb{H}$ to any horizontal strip.

(9) Verify that $f : z \mapsto e^z$ is a conformal map from S to $\mathbb{H}$.

(10) State and prove a reflection principle for a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ which extends continuously to $\overline{\mathbb{D}}$ and maps $\partial\mathbb{D}$ to itself.

(11) Generalize the reflection principle above for a function $f : D(0, r) \to \mathbb{C}$ which extends continuously to $\overline{D(0, r)}$ and maps $\partial D(0, r)$ to $\partial D(0, R)$. Note that reflection on $\partial D(0, r)$ is $z \mapsto r^2/\bar{z}$. What will be the reflection of f here?

(12) State and prove a reflection principle for a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ which extends continuously to $\partial\mathbb{H} = \mathbb{R}$ and maps $\mathbb{R}$ to $\partial\mathbb{D}$. What will be the reflection of f here? Note that the canonical bi-holomorphic function $F : \mathbb{H} \to \mathbb{D}$ fits the hypothesis. What will be its reflection?

(13) Let $f : \mathbb{D} \to \mathbb{C}$ be a function which is holomorphic on $\mathbb{D}$, extends continuously to $\overline{\mathbb{D}}$ and maps ∂D to ∂D. If f does not vanish on $\mathbb{D}$ then show that f is a constant.

(14) Let $f : \mathbb{D} \to \mathbb{C}$ be a meromorphic function on $\mathbb{D}$, extends continuously to $\overline{\mathbb{D}}$ and maps ∂D to ∂D. Show that f is a rational function.

(15) Show that all automorphisms of $\mathbb{H}$ are möbius transformations by matrices of $\mathrm{SL}(2, \mathbb{R})$.

(16) Show that any bi-holomorphic function from $\mathbb{D}$ to $\mathbb{H}$ is of the form $z \mapsto G(Az)$ where $A \in \mathrm{SU}(1,1)$ and Az is the linear fractional transformation given by the matrix A.

(17) Let f be a holomorphic function from $\mathbb{D}$ to itself. Show that

$$|\psi_{f(w_1)}(f(w_2))| \leq \psi_{w_1}(w_2)|, \quad w_1, w_2 \in \mathbb{D}$$

and that equality is achieved when f is an automorphism of $\mathbb{D}$.