1. Questions for practice

Notation: Ω is a connected open subset of $\mathbb{C}$. $f \in H(\Omega)$ means f is holomorphic on Ω. ∂S is the boundary of a subset S of $\mathbb{C}$. $\mathbb{D}$ is unit disc.

(1) Use Rouché's theorem to show a polynomial of degree n has n roots (counting multiplicities).

[WLOG $P(z) = z^n + \{$ all other terms $\}$. Now, z^n has n zeros at 0, but you need to take the disc large enough so that z^n dominates the rest.]

(2) Let $P(z)$ be a polynomial and let D_R be a disc of radius R centered at 0. Show (without assuming FTA) that for positively oriented ∂D_R,

$$\frac{1}{2\pi i}\int_{\partial D_R}\frac{P'(\xi)}{P(\xi)}d\xi$$

converges to a positive integer as $R \to \infty$. This is another proof of FTA.

(3) Find solutions of $e^{z-\alpha} = z$ in $\mathbb{D}$ where $\alpha > 1$. Again use Rouché's theorem, choose suitable f and g.

(4) Show that $2z^5 + 8z - 1$ has no root outside $D(0,2)$. It may follow from straightforward computation, but you should use instead Rouché's theorem.

(5) Let $\mathbb{H}_L = \{z \in \mathbb{C} \mid \Re z < 0\}$ be the left half plane. $f \in H(\mathbb{H}_L)$ and extends continuously to $i\mathbb{R}$. Find condition on f so that f extends as an entire function. (Assume the reflection principle done in class.)

(6) For a function f on the unit circle its Fourier transform $\widehat{f}$ is defined as

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta \ \ \forall n \in \mathbb{Z}.$$

Take $\Omega \supset \overline{\mathbb{D}}$, $F \in H(\Omega)$ and F restricted to $\partial\mathbb{D}$ is f, which is a function on the unit circle. Find relation between $F^{(n)}(0)$ and $\widehat{f}(n)$ when $n \geq 0$.

(7) Let $z_1, z_2 \in \Omega$ be two zeros of f. Show that $f(z) = (z - z_1)^m h(z), z \in \Omega$ for some $m \in \mathbb{N}$ and $h \in H(\Omega)$ such that $h(z_2) = 0$.

(8) Let $z_1, z_2 \in \Omega$ be two poles of f. Show that $f(z) = (z - z_1)^{-m} h(z), z \in \Omega$ for some $m \in \mathbb{N}$ and h is a meromorphic function on Ω with a pole at z_2.

(9) Let $f : \Omega \to \mathbb{C}$ is a continuous function and let D be a disc such that $\overline{D} \subset \Omega$. Suppose that f satisfies Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\xi)}{\xi - z} d\xi, \ \forall z \in D \text{ for positively oriented } \partial D.$$

Then show that f is holomorphic on D.

(10) Let f be an entire function on $\mathbb{C}$. Suppose that for some $R > 0$, $|f(z)| \leq C|z|^{5/2}$ whenever $|z| > R$. Find f up to constants.