Assignment-I, Due on 17/01/25

1. Let $\mathfrak{U}$ be an ultrafilter on X. Define $\mu_{\mathfrak{U}} : 2^X \to \mathbb{R}$, with $2 = \{0, 1\}$ by
$$\mu_{\mathfrak{U}}(A) = \begin{cases} 1 \text{ if } A \in \mathfrak{U} \\ 0 \text{ if } X \setminus A \in \mathfrak{U}. \end{cases}$$
Then $\mu_{\mathfrak{U}}$ is a finitely additive measure. Conversely if $\mu : 2^X \to \{0, 1\} \subseteq \mathbb{R}$ is a $\{0, 1\}$ valued finitely additive measure with $\mu(X) = 1$, then $\mathfrak{U}_\mu = \{A \subseteq X | \mu(A) = 1\}$ is an ultrafilter.

Definition 1.4.3. Let Λ be an index set and $\mathfrak{U}$ be an ultrafilter on Λ. Suppose we have a function $f : \Lambda \to Y$, where Y is a topological space. We say f has $\mathfrak{U}$-limit y for some y in Y if for all $V \in \mathcal{N}_y, f^{-1}(V) \in \mathfrak{U}$. This is denoted by $\mathfrak{U}\text{-}\lim f = y$.

2. Show that if Y is a compact Hausdorff space and $\mathfrak{U}$ is an ultrafilter on Λ then each $f : \Lambda \to Y$ has a $\mathfrak{U}$-limit. Determine $\mathfrak{U}$-limit for a principal ultra filter.

3. (Continued) This allows us to do funny things. For example if $\mathfrak{U}$ is an ultrafilter and $\{x_n\}$ is a bounded sequence in $\mathbb{R}$ then show that we can define $\mathfrak{U}\text{-}\lim x$. Or if we have a sequence $\{s_n\} \subseteq \mathbb{Z}_2$ where $\mathbb{Z}_2$ is the group with two elements with discrete topology. Then show that applying the previous exercise can define $\mathfrak{U}\text{-}\lim s$. Also show if $x \in \mathbb{Z}_2$, then the sequence $\{s'_n := x + s_n\}_n$ satisfies $\mathfrak{U}\text{-}\lim s' = x + \mathfrak{U}\text{-}\lim s$.

4. In a jail the jailer played the following game with the prisoners. All the prisoners were given T-shirts with a tick or a cross in the backside of the T-shirt and were arranged in a queue so that any prisoner could see the backsides of all the prisoners standing in front of him/her. Based on that he/she has to guess the mark on the T-shirt he/she is wearing. Let us assume prisoners are standing in positions 1,2, etc. so that the prisoner standing on position i can see the backsides of prisoners $i + 1, i + 2, \ldots$ etc. After the first prisoner declares his mark the second prisoner has to declare and so on. If everybody except the first prisoner can answer correctly then they will be released. Device a winning strategy.

5. (Stone-Cech Compactification) Let X be a discrete set. Then a filter is a subset of $\mathcal{P}(X) = 2^X$ or an element of 2^{2^X}. Therefore βX, the set of ultrafilters on X is a subset of 2^{2^X}. By Tychonoff's theorem 2^{2^X} is compact. Show that βX is closed and therefore compact.

6. (Continued) Consider the map $\mathfrak{U} : X \ni x \mapsto \mathfrak{U}_x \in \beta X$, where $\mathfrak{U}_x$ is the principal ultrafilter determined by x. Since $\mathfrak{U}$ is one to one we can and we will identify X as a subset of βX. Show that $\mathfrak{U}(X)$ is dense in βX.

7. (Continued) Finally, given any compact Hausdorff space Y and a map $f : X \to Y$ define $\tilde{f} : \beta X \to Y$ by $\tilde{f}(\mathfrak{U}) = \mathfrak{U}\text{-}\lim f$. Show that $\tilde{f}$ is continuous.