Functional Analysis

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Preface

These are my planned course notes on Functional Analysis offered to MMath 1st year students.

Week 0

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0.1 Things we need to decide

Already decided marks distribution.

0.2 What do we do in this course?

Linear algebra coupled with topology.

Week 1

Normed Linear Spaces

Definition 1.0.1. Let E be a real or complex vector space. A norm on E is a function $\|\cdot\| : E \to \mathbb{R}$ such that

1. $\forall v \in E, \|v\| \geqslant 0$ with equality iff $v = 0$.
2. $\forall v \in E, \forall \lambda \in \mathbb{R}, \|\lambda v\| = |\lambda|\|v\|$.
3. $\forall v, w \in E, \|v+w\| \leqslant \|v\|+\|w\|$. This property is also called triangle inequality.

Definition 1.0.2 (Normed Linear Space)**.** A Normed Linear Space $(V, \|\cdot\|)$ consists of a real or complex vector space V along with a norm $\|\cdot\|$ on V. If the norm is clear from the context we may drop it from notation.

Exercise 1.0.3. Let $(V, \|\cdot\|)$ be a Normed Linear Space. Then $d_{\|\cdot\|}(v, w) := \|v - w\|$ is a metric called the metric induced by the norm.

Proposition 1.0.4. Let $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$ be normed linear spaces. A linear map $T : E \to F$ is continuous iff it is continuous at 0.

Proof. Only if part is obvious we only need to show the if part. To establish continuity at w, given $\epsilon > 0$ we need to find $\delta > 0$ such that $d_{\|\cdot\|_E}(w, w') < \delta \implies d_{\|\cdot\|_F}(T(w), T(w')) < \epsilon$. Since T is continuous at zero given $\epsilon > 0$, there exists $\delta > 0$ such that $\|T(v)\|_F < \epsilon$ whenever $\|v\|_E < \delta$. This δ works because if $\delta > d_{\|\cdot\|_E}(w, w') = \|w - w'\|_E$ then

$$\begin{aligned} \epsilon &> \|T(w) - T(w')\|_F \\ &= d_{\|\cdot\|_F}(T(w), T(w')). \end{aligned}$$

□

Proposition 1.0.5. Let $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$ be normed linear spaces. A linear map $T : E \to F$ is continuous iff there exists $C > 0$ such that $\|T(v)\|_F \leqslant C\|v\|_E, \forall v \in E$.

Proof. If part: We know it is enough to show continuity at zero and that follows because $\|v\|_E < \epsilon/C \implies \|T(v)\|_F < \epsilon$.

Only if part: Since T is continuous at zero we know $\exists \delta > 0$ such that $(*)$ $\|v\|_E < \delta \implies \|T(v)\|_F < 1$.

Claim: $\|T(v)\|_F \leqslant \frac{2}{\delta}\|v\|_E$.

If not then $\exists w$ such that $\|T(w)\|_F > \frac{2}{\delta}\|w\|_E$. Let $w' = \frac{2\delta}{3\|w\|_E}w$. Then $\|w'\|_E = \frac{2\delta}{3} < \delta$. Therefore by $(*)$

$$\begin{aligned} 1 > \|T(w')\|_F &= \frac{2\delta}{3\|w\|_E}\|T(w)\|_F \\ &> \frac{2\delta}{3\|w\|_E}\frac{2\|w\|_E}{\delta} > 1, \end{aligned}$$

a contradiction! □

Definition 1.0.6. Let $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$ be normed linear spaces. A linear map $T : E \to F$ is said to be bounded if there exists $C > 0$ such that $\forall v \in E, \|T(v)\|_F \leqslant C\|v\|_E$. The set of bounded linear maps from E to F is denoted by $\mathcal{L}(E, F)$ and $\mathcal{L}(E, E)$ is denoted by $\mathcal{L}(E)$. In general $\mathcal{L}(E, F)$ is a subset of $L(E, F)$.

Definition 1.0.7 (The Dual of a Normed Linear Space)**.** Let E be a normed linear space over the field $\mathbb{K}$ where $\mathbb{K}$ could be $\mathbb{R}$ or $\mathbb{C}$. Then the space $\mathcal{L}(E, \mathbb{K})$ of bounded linear functionals is called the dual space of E and is denoted by E^*.

1.1 Hahn Banach Theorems: Analytic Forms

Theorem 1.1.1 (Hahn Banach: Analytic Form)**.** *Let E be a real normed linear space and $F \subseteq E$ be a subspace. Let $\phi \in F^*$. Then there exists $\tilde{\phi} \in E^*$ such that $\|\tilde{\phi}\| = \|\phi\|$.*

Proof. Step 1: Let $F_1 = F + \mathbb{R}x_0$, where $x_0 \in E \setminus F$. Let us denote a prospective candidate for $\tilde{\phi}(x_0)$ by ϕ_0. Then we must have

$$|\phi(x) + \lambda\phi_0| \leqslant \|\phi\|\|x + \lambda x_0\|, \forall x \in F, \lambda \in \mathbb{R}. \tag{1.1}$$

These inequalities are equivalent to the following system of inequalities.

$$\phi(x) + \lambda\phi_0 \leqslant \|\phi\|\|x + \lambda x_0\|, \forall x \in F, \lambda \in \mathbb{R} \tag{1.2}$$
$$-(\phi(x) + \lambda\phi_0) \leqslant \|\phi\|\|x + \lambda x_0\|, \forall x \in F, \lambda \in \mathbb{R} \tag{1.3}$$

Since $-(\phi(x)+\lambda\phi_0) = \phi(-x)+(-\lambda)\phi_0$ and $\|x+\lambda x_0\| = \|-(x+\lambda x_0)\|$ the system of inequalities given by (1.3) and (1.2) are equivalent. So, we can say that the system of inequalities given by (1.1) is equivalent with (1.2). Considering the cases $\lambda \lessgtr 0$ in (1.2) we get

$$\phi(y) - \|\phi\|\|y - x_0\| \leqslant \phi_0 \leqslant \|\phi\|\|x + x_0\| - \phi(x), \forall x, y \in F \tag{1.4}$$

So, we must show $\sup_{y\in F}(\phi(y)-\|\phi\|\|y-x_0\|) \leqslant \inf_{x\in F}(\|\phi\|\|x+x_0\|-\phi(x))$ or equivalently

$$\phi(y)-\|\phi\|\|y-x_0\| \leqslant \|\phi\|\|x+x_0\|-\phi(x), \forall x,y \in F. \tag{1.5}$$

But this follows from $\phi(x)+\phi(y)=\phi(x+y) \leqslant \|\phi(x+y)\| \leqslant \|\phi\|\|x+y\| \leqslant \|\phi\|(\|x+x_0\|+\|y-x_0\|)$ and we can take any element from the closed interval $[\sup_{y\in F}(\phi(y)-\|\phi\|\|y-x_0\|), \inf_{x\in F}(\|\phi\|\|x+x_0\|-\phi(x))]$ as ϕ_0. Thus we have established the existence of an extension ϕ_1 of ϕ to F_1. Also from (1.1) we conclude that $\|\phi_1(x)\| \leqslant \|\phi\|\|x\|, \forall x \in F_1$. In other words $\|\phi_1\| \leqslant \|\phi\|$. Also $\|\phi_1\| = \sup_{x\in F_1:\|x\|=1}\|\phi_1(x)\| \geqslant \sup_{x\in F:\|x\|=1}\|\phi_1(x)\| = \|\phi\|$. Therefore ϕ_1 is a norm preserving extension of ϕ.

Step 2: Let $\mathcal{P} = \{(F_1,\phi_1) : F \subseteq F_1, \phi_1 \in F_1^*, \phi_1|_F = \phi, \|\phi_1\| = \|\phi\|\}$. This is a POset with partial order given by $(F_1',\phi_1') \succeq (F_1,\phi_1)$. Every chain in $\mathcal{P}$ has an upper bound and therefore by Zorn's lemma $\mathcal{P}$ has a maximal element, say $(\tilde{F},\tilde{\phi})$. We claim that $\tilde{F}$ must be E else by applying step 1 to $\tilde{F}$ we can obtain a further extension contradicting the maximality. □

An analysis of the argument: First thing to note is, in the above argument step 1 is the real step. If you notice the argument carefully you will see whenever we have used $\|\cdot\|$ it is actually $\|\phi\|\|\cdot\|$. So, it makes sense to rewrite the argument using the notation $p(x)=\|\phi\|\|x\|$ and observe which properties of this function actually goes into the argument.

Proof of 1.1.1 in new notation. Step 1: Let $F_1 = F+\mathbb{R}x_0$, where $x_0 \in E\setminus F$. Let us denote a prospective candidate for $\tilde{\phi}(x_0)$ by ϕ_0. Then we must have

$$\phi(x)+\lambda\phi_0 \leqslant p(x+\lambda x_0), \forall x \in F, \lambda \in \mathbb{R}. \tag{1.2'}$$

Considering the cases $\lambda \lesseqgtr 0$ in (1.2′) we get

$$\phi(y)-p(y-x_0) \leqslant \phi_0 \leqslant p(x+x_0)-\phi(x), \forall x,y \in F \tag{1.4'}$$

Here we have used one property of the function p, called positive homogeneity, meaning $p(\lambda x)=\lambda p(x), \forall \lambda>0, x \in E$. To show 1.4′ we must show that $\sup_{y\in F}(\phi(y)-p(y-x_0)) \leqslant \inf_{x\in F}(p(x+x_0)-\phi(x))$ or equivalently

$$\phi(y)-p(y-x_0) \leqslant p(x+x_0)-\phi(x), \forall x,y \in F. \tag{1.5'}$$

But this follows from $\phi(x)+\phi(y)=\phi(x+y) \leqslant p(x+y) \leqslant p(x+x_0)+p(y-x_0)$ because p satisfies triangle inequality and we can take any element from the interval $[\sup_{y\in F}(\phi(y)-p(y-x_0)), \inf_{x\in F}(p(x+x_0)-\phi(x))]$ as ϕ_0. Thus we have established the existence of an extension ϕ_1 of ϕ to F_1. Also from (1.2′) we conclude that $\phi_1(x) \leqslant p(x), \forall x \in F_1$.

Step 2: Let $\mathcal{P} = \{(F_1,\phi_1) : F \subseteq F_1, \phi_1 \in F_1^*, \phi_1|_F = \phi, \phi_1(x) \leqslant p(x), \forall x \in F_1\}$. This is a POset with partial order given by $(F_1',\phi_1') \succeq (F_1,\phi_1)$. Every chain in $\mathcal{P}$ has an upper bound and therefore by Zorn's lemma $\mathcal{P}$ has a maximal element, say $(\tilde{F},\tilde{\phi})$. We claim that $\tilde{F}$ must be E else by applying step 1 to $\tilde{F}$ we can obtain a further extension contradicting the maximality. □

Let us note that we have used two properties of the function p and those are

1. Triangle inequality/ subadditivity: $p(x+y) \leqslant p(x) + p(y)$.
2. Positive homogeneity: $p(\lambda x) = \lambda p(x), \forall \lambda \in \mathbb{R}_{>0}, x \in E$.

Look and behold just by changing notation we have proved.

Theorem 1.1.2 (Hahn Banach, analytic version 2)**.** *Let E be a real vector space and $p : E \longrightarrow \mathbb{R}$ a positively homogeneous subadditive function. Let $F \subseteq E$ be a subspace and $\phi : F \longrightarrow \mathbb{R}$ a linear map satisfying $\phi(x) \leqslant p(x), \forall x \in F$. Then ϕ admits an extension $\tilde{\phi}$ to E satisfying $\tilde{\phi}(x) \leqslant p(x), \forall x \in E$.*

In fact this is stronger than theorem (1.1.1) in the sense that it implies theorem (1.1.1). Let us see that.

Proof of theorem (1.1.1) using theorem(1.1.2). Let $p(x) = \|\phi\|\|x\|$. Then $\phi(x) \leqslant |\phi(x)| \leqslant p(x), \forall x \in F$. By theorem (1.1.2) we get an extension $\tilde{\phi}$ of ϕ such that $\tilde{\phi}(x) \leqslant p(x), \forall x \in E$. Note that $-\tilde{\phi}(x) = \tilde{\phi}(-x) \leqslant p(-x) = p(x), \forall x \in E$. Therefore $|\tilde{\phi}(x)| \leqslant p(x) = \|\phi\|\|x\|$. In other words $\|\tilde{\phi}\| \leqslant \|\phi\|$. The other inequality required to show $\|\tilde{\phi}\| = \|\phi\|$ follows from

$$\|\tilde{\phi}\| = \sup_{x:x\in E, \|x\|\leqslant 1} |\tilde{\phi}(x)| \geqslant \sup_{x:x\in F, \|x\|\leqslant 1} |\tilde{\phi}(x)| = \sup_{x:x\in F, \|x\|\leqslant 1} |\phi(x)| = \|\phi\|. \qquad \square$$

Is this version/generalisation of any use? This question could be annoying but we won't hesitate to ask this. Later we will define topologies on vector spaces using seminorms. Those will be locally convex spaces. Using this version we can show existence of continuous linear functionals on locally convex spaces. But before doing any of that let us obtain versions of these results in the complex case. To be able to apply this result to complex vector spaces we need a simple observation that relates a complex linear functional with its real part, a real linear map.

Lemma 1.1.3. *Let E be a vector space over $\mathbb{C}$.*

(i) If $f : E \longrightarrow \mathbb{R}$ is an $\mathbb{R}$ linear functional, then $\tilde{f}(x) = f(x) - if(ix)$ is a $\mathbb{C}$ linear functional and $f = \Re\tilde{f}$.

(ii) If $g : E \longrightarrow \mathbb{C}$ is $\mathbb{C}$ linear $f = \Re g$ and $\tilde{f}$ is defined as above then $\tilde{f} = g$.

(iii) If E is a normed space and $f, \tilde{f}$ are as in (i) then $\|f\| = \|\tilde{f}\|$.

Proof. (iii) Suppose $|\tilde{f}(x)| \leqslant \|\tilde{f}\|\|x\|$, then

$$f(x) = \Re\tilde{f}(x) \leqslant |\tilde{f}(x)| \leqslant \|\tilde{f}\|\|x\|.$$

Also

$$-f(x) = \Re\tilde{f}(-x) \leqslant |\tilde{f}(-x)| \leqslant \|\tilde{f}\|\|x\|.$$

Hence $|f(x)| \leqslant \|\tilde{f}\|\|x\|$.

Now assume $|f(x)| \leqslant \|f\|\|x\|$. Choose θ such that $\tilde{f}(x) = e^{i\theta}|\tilde{f}(x)|$. Hence

$$|\tilde{f}(x)| = \tilde{f}(e^{-i\theta}x) = \Re\tilde{f}(e^{-i\theta}x) = f(e^{-i\theta}x) \leqslant \|f\|\|e^{-i\theta}x\|.$$ □

Definition 1.1.4. A real valued sub-additive function p defined on a vector space E is called a seminorm if $p(\alpha \cdot x) = |\alpha| p(x), \forall \alpha \in \mathbb{K}, x \in E$.

Lemma 1.1.5. *Let p be a seminorm on a vector space E, then (a) $p(0) = 0$; (b) $|p(x) - p(y)| \leqslant p(x-y), \forall x, y \in E$; (c) $p(x) \geqslant 0$.*

Proof. (a) This follows from, $p(0) = p(0 \cdot x) = |0| \cdot p(x) = 0$.
(b) Note that

$$p(x) - p(y) = p(x - y + y) - p(y) \leqslant p(x-y) + p(y) - p(y) = p(x-y).$$

Interchanging x and y we obtain the other inequality $p(y) - p(x) \leqslant p(x-y)$ needed to complete the proof.
(c) We have $p(x) = p(x-0) \geqslant |p(x) - p(0)| = |p(x)| \geqslant 0$. □

Theorem 1.1.6. *Suppose E is a subspace of a vector space F, p is a seminorm on F and $\phi : E \to \mathbb{K}$ a linear map such that $|\phi(x)| \leqslant p(x), \forall x \in E$. Then there is a linear functional $\tilde{\phi}$ defined on F such that $\tilde{\phi}|_E = \phi$ and $|\tilde{\phi}(x)| \leqslant p(x)$.*

Proof. Case 1 ($\mathbb{K} = \mathbb{R}$): We have $p(-x) = p(x)$ and we are done by theorem (1.1.2).

Case 2 ($\mathbb{K} = \mathbb{C}$): Let $\phi_1 = \Re\phi$, then there exists real linear $\tilde{\phi_1}$ on F such that $\tilde{\phi_1}|_E = \phi_1$ Let $\tilde{\phi}(x) = \tilde{\phi_1}(x) - i\tilde{\phi_1}(ix)$, then $\tilde{\phi}|_E = \phi$. Finally given any $x \in F, \exists \lambda \in \mathbb{C}$ such that $|\lambda| = 1, \lambda\tilde{\phi}(x) = |\tilde{\phi}(x)|$. We have,

$$|\tilde{\phi}(x)| = \tilde{\phi}(\lambda x) = \tilde{\phi_1}(\lambda x) \leqslant p(\lambda x) = p(x).$$

□

Corollary 1.1.7 (Hahn-Banach Theorem)**.** Let $E \subseteq F$ be normed linear spaces and $\phi : E \to \mathbb{C}$ a continuous linear functional then there exists a continuous linear functional $\tilde{\phi} : F \to \mathbb{C}$ such that $\tilde{\phi}|_E = \phi$ and $\|\tilde{\phi}\| = \|\phi\|$.

Proof. Take $p(x) = \|\phi\|\|x\|$ and obtain $\tilde{\phi}$ such that $|\tilde{\phi}| \leqslant p(x)$. This means $\|\tilde{\phi}\| \leqslant \|\phi\|$. We have argued the other inequality required to prove equality several times. □

Week 2

Applications of Hahn-Banach theorem

Corollary 2.0.1 (Corollary to Hahn-Banach Theorem)**.** Let E be a normed linear space and $x \in E$. Then there exists $x^* \in E^*$ such that $x^*(x) = \|x\|, \|x^*\| = 1$.

Proof. Let F be the span of x and $\phi : F \to \mathbb{K}$ be the linear functional given by $\phi(\lambda x) = \lambda\|x\|, \forall \lambda \in \mathbb{K}$. Then $\|\phi\| = 1$. Let x^* be a norm preserving extension of ϕ. □

Corollary 2.0.2 (Corollary to Hahn-Banach Theorem)**.** Let E be a normed linear space and E^* it's dual. Then the norm of $x \in E$ satisfies,

$$\|x\| = \sup\{| < x^*, x > | : \|x^*\| \leqslant 1\},$$

where $< x^*, x >$ denotes $x^*(x)$.

Proof. Let $x \in E$, then for any $x^* \in E^*$ with $\|x^*\| \leqslant 1$, we have $| < x^*, x > | \leqslant \|x^*\|\|x\| \leqslant \|x\|$. This shows that

$$\|x\| \leqslant \sup\{| < x^*, x > | : \|x^*\| \leqslant 1\}.$$

For the other inequality using the Hahn Banach theorem obtain x^* of norm one such that $x^*(x) = \|x\|$. □

Now that we have shown that E^* is a nontrivial space it makes sense to recognise one crucial property enjoyed by duals of normed linear spaces, namely completeness. Stefan Banach initiated systematic study of these spaces and he called them B spaces. Frechet started calling them Banach spaces. Let us officially record the definition.

Definition 2.0.3 (Banach Space)**.** A complete normed linear space is called a Banach space

Proposition 2.0.4. Let E be a normed linear space and F be a Banach space. Then $\mathcal{L}(E, F)$ is a Banach space. In particular E^* is a Banach space.

Proof. Let $\{T_n\}$ be a Cauchy sequence in $\mathcal{L}(E,F)$. Then $\forall \epsilon > 0, \exists N$ such that $\|T_n - T_m\| < \epsilon, \forall n, m \geqslant N$. Then for any $x \in E$,

$$\|T_n x - T_m x\| < \epsilon \|x\| \text{ for } n, m \geqslant N. \tag{2.1}$$

Using completeness of F we get $\lim T_n x = Tx$. Also

$$T(\alpha x + \beta y) = \lim T_n(\alpha x + \beta y) = \lim \alpha T_n(x) + \beta T_n(y) = \alpha T(x) + \beta T(y).$$

Therefore T is linear and it is bounded because

$$\|T(x)\| = \lim \|T_n(x)\| = \lim \|T_N(x) + (T_n(x) - T_N(x))\| \leqslant (\epsilon + \|T_N\|)\|x\|.$$

Letting m tend to infinity in (2.1) we get $\|T_n - T\| \leqslant \epsilon, \forall n > N$. Thus $T = \lim T_n \in \mathcal{L}(E,F)$ showing completeness of $\mathcal{L}(E,F)$. □

Proposition 2.0.5. Let E be a Banach space. A subspace $F \subseteq E$ is complete iff it is closed.

Proof. If part: Let $\{x_n\} \subseteq F$ be a Cauchy sequence. Then using completeness of E we know $\lim x_n = x$ for some $x \in E$. Since F is closed $\lim x_n = x \in F$. Thus F is complete.

Only if part: Let $\{x_n\} \subseteq F$ be converging to x. As F is complete $x \in F$. Therefore F is closed. □

Exercise 2.0.6. Show that a finite dimensional subspace of a normed linear space is always closed. Hint: Any two norms on a finite dimensional space are equivalent.

2.1 Canonical embedding into second dual

Definition 2.1.1. Let $j_E : E \to E^{**}$ be the map defined by $j_E(x)(x^*) =< x^*, x >$. Then

$$\|j_E(x)\| = \sup_{x^*:\|x^*\|=1} | < x^*, x > | = \|x\|.$$

Therefore j_E is an isometric embedding of E into E^{**}, often referred as the canonical embedding of E into E^{**}. The norm closure of $j_E(E)$ is the completion of E. We say E is reflexive if j is an isomorphism.

Proposition 2.1.2. Let E be a normed linear space. Then the completion of E is a Banach space.

Proof. The norm closure of $j_E(E)$ is the completion of E. Being closure of a subspace it is a complete normed linear space or which is same as a Banach space. □

Remark 2.1.3. Can there be a non-reflexive normed linear space E such that there is an isometric isomorphism $T \in \mathcal{L}(E, E^{**})$, i.e., an isomorphism T satisfying $\|T(x)\| = \|x\|, \forall x \in E$? A counter example was given by Robert James. It is in his honour we denote the canonical embedding by j.

Definition/Proposition 2.1.4. Let E, F be Banach spaces and $T \in L(E, F)$. Then $T^* : F^* \to E^*$ defined by $T^*(\phi)(x) = (\phi \circ T)(x)$ defines a bounded linear map, called the adjoint of T with $\|T^*\| = \|T\|$. Also $I_E^* = I_{E^*}$, where I_E, I_{E^*} be the identity mappings of E, E^* respectively. If $S \in \mathcal{L}(F, G)$ then $(S \circ T)^* = T^* \circ S^*$.

Proof. Let $\phi \in F^*$ then

$$\begin{aligned} \|T^*(\phi)\| &= \sup\{|T^*(\phi)(x)| : x \in E, \|x\| \leqslant 1\} \\ &= \sup\{|\phi(T(x))| : x \in E, \|x\| \leqslant 1\} \\ &\leqslant \|\phi\|\|T\|. \end{aligned}$$

Therefore $\|T^*\| \leqslant \|T\|$. We give two proofs of the other inequality $\|T\| \leqslant \|T^*\|$.

First proof.

$$\begin{aligned} \|T\| &= \sup\{\|T(x)\| : x \in E, \|x\| \leqslant 1\} \\ &= \sup\{|\phi(T(x))| : x \in E, \phi \in F^*, \|x\|, \|\phi\| \leqslant 1\} \\ &\leqslant \sup\{\|T^*(\phi)\| : \phi \in F^*, \|\phi\| \leqslant 1\} \\ &\leqslant \|T^*\|. \end{aligned}$$ □

Second proof. Let $x \in E, \phi \in F^*$. Then we have

$$T^{**}(j_E(x))(\phi) = j_E(x)(T^*\phi) = T^*(\phi)(x) = \phi(T(x)) = j_F(T(x))(\phi).$$

In other words

$$T^{**} \circ j_E = j_F \circ T. \tag{2.2}$$

In categorical parlance this means j is a natural transformation. (Soon we will elaborate on this.) Therefore,

$$\|T\| = \sup_{x \in B_E} \|T(x)\| = \sup_{x \in B_E} \|j(T(x))\| = \sup_{x \in B_E} \|T^{**}(j(x))\| \leqslant \sup_{x^{**} \in B_{E^{**}}} \|T^{**}(x^{**})\| = \|T^{**}\|$$

Using $\|T^*\| \leqslant \|T\|$ for T^* we get $\|T^{**}\| \leqslant \|T^*\|$. Thus $\|T\| \leqslant \|T^*\|$. □

Let us look back and reflect on what have we done just now. To any normed linear space E we have associated a normed linear space, namely E^*. Also to any $T \in \mathcal{L}(E, F)$ we have associated a $T^* \in \mathcal{L}(F^*, E^*)$. This association satisfies two more properties, (i) $I_E^* = I_{E^*}$ and (ii) $S \in \mathcal{L}(F, G)$ then $(S \circ T)^* = T^* \circ G^*$. Now in mathematics whenever some structure occurs frequently we introduce terminology so that we can talk about the structure and investigate its properties. In this case the relevant structure is of categories and functors.

2.2 Categories and functors

Definition 2.2.1 (Locally small category)**.** A locally small category $\mathcal{C}$ consists of a class $\mathrm{Ob}(\mathcal{C})$ called objects of $\mathcal{C}$ and given any two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Mor}_{\mathcal{C}}(A, B)$ called morphisms of $\mathcal{C}$. When there is no scope for confusion we will drop $\mathcal{C}$ from the notation $\mathrm{Mor}_{\mathcal{C}}$. If $f \in \mathrm{Mor}(A, B)$, then we may also write $f : A \to B$ or $A \xrightarrow{f} B$. We will denote $\mathrm{Mor}(A, A)$ by $\mathrm{Mor}(A)$. Given $A, B, C \in \mathrm{Ob}(\mathcal{C})$, there is a map $\circ : \mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$ called composition and for each $A \in \mathrm{Ob}(\mathcal{C})$ a morphism $I_A \in \mathrm{Mor}(A)$, called the identity morphism of A such that $\forall f \in \mathrm{Mor}(A, B), g \in \mathrm{Mor}(B, C), \forall h \in \mathrm{Mor}(C, D)$ we have $\circ(\circ(f, g), h) = \circ(f, \circ(g, h))$ and $\circ(I_A, f) = f = \circ(f, I_B)$. We denote $\circ(f, g)$ by $g \circ f$. In this notation the conditions become associativity $h \circ (g \circ f) = (h \circ g) \circ f$ and $f \circ I_A = f = I_B \circ f$.

Example 2.2.2. The category $\mathcal{S}ets$ has sets as objects and functions as morphisms.

Example 2.2.3. The category $\mathcal{G}p$ has groups as objects and group homomorphisms as morphisms. The usual composition of functions define composition.

Example 2.2.4. Let G be a group. Then we can define a category with only one object $*$ and $\mathrm{Mor}(*) = G$. The identity element of G plays the role of I_* while the group multiplication defines the composition. This example shows morphisms may not be functions. Also in a sense the notion of category generalises the notion of groups.

Example 2.2.5. The category $\mathcal{N}ls_{\mathbb{K}}$ the category of normed linear spaces over $\mathbb{K}$ has normed $\mathbb{K}$ vector spaces as objects and bounded linear maps as mrphisms.

Example 2.2.6. The category $\mathcal{B}an$ has Banach spaces as objects with $\mathrm{Mor}(E, F) = \mathcal{L}(E, F)$.

Example 2.2.7. The category $\mathcal{B}an_1$ has Banach spaces as objects with $\mathrm{Mor}(E, F) = \{T \in \mathcal{L}(E, F) : \|T\| \leqslant 1\}$.

Definition 2.2.8. Let $\mathcal{C}, \mathcal{D}$ be categories. A covariant (contravariant) functor $F : \mathcal{C} \to \mathcal{D}$ associates to an object $A \in \mathrm{Ob}(\mathcal{C})$ an object $F(A) \in \mathrm{Ob}(\mathcal{D})$ and to a morphism $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ an element $F(f) \in \mathrm{Mor}_{\mathcal{D}}(F(A), F(B))(F(f) \in \mathrm{Mor}_{\mathcal{D}}(F(B), F(A))$ such that

1. For all f, g so that the composition $g \circ f$ is defined we have $F(g) \circ F(f) = F(g \circ f)(F(f) \circ F(g) = F(g \circ f))$.

2. For all $A \in \mathrm{Ob}(\mathcal{C}), F(I_A) = I_{F(A)}$.

Covariant functors are often called functors.

In this terminology we can state what we have already proved.

Example 2.2.9. The dualization functor $* : \mathcal{N}ls_{\mathbb{K}} \to \mathcal{N}ls_{\mathbb{K}}$ is the contravariant functor sending $E \in \mathrm{Ob}(\mathcal{N}ls_{\mathbb{K}})$ to E^* and $T \in \mathcal{L}(E, F)$ to T^*. Since dualization is contravariant applying it twice we get the covariant functor second dual.

Definition 2.2.10. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. Then a natural transformation $\eta : F \to G$ associates a morphism $\eta_A \in \mathrm{Mor}_{\mathcal{D}}(F(A), G(A))$ for each object A of $\mathcal{C}$ so that for each $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ we have $\eta_B \circ F(f) = G(f) \circ \eta_A$. This is also expressed by saying the following diagram commutes.

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\eta_B} & G(B) \end{array}$$

Example 2.2.11. The James map gives a natural transformation $j : \mathrm{Id} \to **$. We have verified the relevant condition in (2.2).

2.3 Reflexive Banach spaces

Proposition 2.3.1. A closed subspace of a reflexive Banach space is reflexive.

Proof. Let $F \subseteq E$ be a closed subspace with $i : F \hookrightarrow E$ the inclusion map. Let $y^{**} \in F^{**}$. We have to exhibit $y \in F$ such that $j_F(y) = y^{**}$. Since E is reflexive there is $x \in E$ such that $i^{**}(y^{**}) = j_E(x)$. It is enough to show that $x \in F$. In other words $i(x) = x$. Because then $i^{**}(y^{**}) = j_E(x) = j_E \circ i(x) = i^{**}(j_F(x))$. If we can show i^{**} is one to one then we will get $y^{**} = j_F(x)$. So we need to show two things, (i) $x \in F$ and i^{**} is one to one.

Proof of $x \in F$. Suppose $x \notin F$. Then by Hahn-Banach there exists $x^* \in E^*$ such that $x^*(F) = 0$ or equivalently $i^*(x^*) = 0$ and $x^*(x) = 1$. We have the following chain of equalities

$$1 = \langle x^*, x \rangle = \langle j_E(x), x^* \rangle = \langle i^{**}(y^{**}), x^* \rangle = \langle y^{**}, i^*(x^*) \rangle = 0!$$

This contradiction shows $x \in F$. □

Injectivity of i^{**}.Let $y^* \in F^*$ be arbitrary and x^* be a norm preserving extension of y^*, in other words $\langle x^*, i(y) \rangle = \langle y^*, y \rangle, \forall y \in F$. So, $\langle i^*(x^*) - y^*, y \rangle = \langle x^*, i(y) \rangle - \langle y^*, y \rangle = 0, \forall y \in F$. Thus $y^* = i^*(x^*)$. In other words i^* is onto. Suppose $i^{**}(z^{**}) = 0$ for some $z^{**} \in F^{**}$. Then for all $x^* \in E^*$ we have $\langle z^{**}, i^*(x^*) \rangle = 0$. Since i^* is onto, this means $z^{**} = 0$ □

Proposition 2.3.2. Let E be a Banach space. Then E is reflexive iff E^* is reflexive.

Proof. Only if part: Let E be reflexive. We have to show every $x^{***} \in E^{***}$ is of the form $j_{E^*}(x^*)$. So, given x^{***} define x^* by

$$\langle x^*, x\rangle = \langle x^{***}, j_E(x)\rangle, \forall x \in E. \tag{2.3}$$

Claim: $j_{E^*}(x^*) = x^{***}$.

Proof of claim. We have to show $\langle x^{***}, x^{**}\rangle = \langle j_{E^*}(x^*), x^{**}\rangle, \forall x^{**} \in E^{**}$. So, let $x^{**} \in E^{**}$ be arbitrary. Then using reflexivity of E we get $x^{**} = j_E(x)$ for some $x \in E$. The following chain of equalities

$$\langle j_{E^*}(x^*), x^{**}\rangle = \langle x^{**}, x^*\rangle = \langle j_E(x), x^*\rangle = \langle x^{***}, j_E(x)\rangle = \langle x^{***}, x^{**}\rangle$$

show $x^{***} = j_{E^*}(x^*)$. □

If part: If E^* is reflexive then by the only if part E^{**} is reflexive. By proposition (2.3.1), $j_E(E)$ is reflexive. Therefore so is E. □

Proposition 2.3.3. Let E, F be isomorphic Banach spaces. Then E is reflexive iff F is reflexive

Proof. It is enough to show one of the implications because the other follows by symmetry. We will show the only if part. Let $T : E \to F$ be an isomorphism. Then $T^{**} : E^{**} \to F^{**}$ is an isomorphism. Since James map is a natural transformation we have $T^{**} \circ j_E = j_F \circ T$. The left hand side is surjective because E is reflexive. Therefore the right hand side must be surjective as well. Since T is an isomorphism this implies j_F is surjective. □

2.4 Duals of some Banach spaces

Proposition 2.4.1. Let $c_0 = \{\{x_n\} \subseteq \mathbb{R} : \lim x_n = 0\}$ be the space of sequences of real numbers converging to zero. This is a Banach space with sup-norm. The dual of c_0 is linearly isometrically isomorphic with ℓ_1.

Proof. Let $e^{(n)} = \{e^{(n)}_k\} \in c_0$ where $e^{(n)}_k = \delta_{nk}$ is the Kronecker delta. These $e^{(n)}$'s do not form a Hamel basis however an arbitrary $\mathbf{x} = \{x_n\} \in c_0$ can always be expressed as $\mathbf{x} = \sum_{n=1}^{\infty} x_n e^{(n)} = \lim_k \sum_{n=1}^{k} x_n e^{(n)}$. Here the limit converges in the topology of c_0. Any bounded linear functional $\phi \in c_0^*$ satisfies

$$\phi(\mathbf{x}) = \lim_k \sum_{n=1}^{k} x_n \phi(e^{(n)}) = \lim_k \sum_{n=1}^{k} x_n \phi_n$$

where $\phi_n = \phi(e^{(n)})$.

Claim: $\{\phi_n\} \in \ell_1, \|\phi\| = \sum_n |\phi_n|$.

Proof of claim. We denote by sgn the signum function given by $\text{sgn}(x) = 1$ for $x \geqslant 0$, $\text{sgn}(x) = -1$ for $x < 0$. For each N let $x^{(N)} \in B_{c_0}$, the unit ball of c_0, be the sequence

$$x_n^{(N)} = \begin{cases} \text{sgn}(\phi_n) \text{ if } 1 \leqslant n \leqslant N \\ 0 \text{ otherwise} \end{cases}.$$

Then for all N we have

$$\sum_{n=1}^{N} |\phi_n| = \phi(x^{(N)}) = |\phi(x^{(N)}| \leqslant \|\phi\| \|x^{(N)}\| = \|\phi\|.$$

Since this happens for all N we get $\{\phi_n\} \in \ell_1$ and $\sum_n |\phi_n| \leqslant \|\phi\|$. On the other hand

$$|\phi(\mathbf{x})| = |\lim_k \sum_{n=1}^{k} x_n \phi_n| \leqslant \lim \sum_{n=1}^{k} |x_n||\phi_n| \leqslant \|\mathbf{x}\| \sum_n |\phi_n|$$

shown $\|\phi\| \leqslant \sum |\phi_n|$ the other inequality required to show $\|\phi\| = \sum |\phi_n|$. □

We have established a linear isometry $\Phi : c^* \ni \phi \mapsto \{\phi_n\} \in \ell_1$. Only thing that remains to be shown is this is onto. But that is obvious because given any $\{\phi_n\} \in \ell_1$ we can define $\phi \in c^*$ as $\phi(\mathbf{x}) = (\lim x_n)\phi_0 + \sum_{n=1}^{\infty} x_n \phi_n$. This series converges because $\{x_n\}$ is bounded and $\sum |\phi_n| < \infty$. Clearly $\Phi(\phi) = \{\phi_n\}$. □

Proposition 2.4.2. Let $c = \{\{x_n\} \subseteq \mathbb{R} : \lim x_n \text{ exists}\}$ be the space of convergent sequences of real numbers. This is a Banach space with sup-norm. The dual of c is linearly isometrically isomorphic with ℓ_1.

Proof. Let $e^{(n)} = \{e_k^{(n)}\} \in c_0$ where $e_k^{(n)} = \delta_{nk}$ is the Kronecker delta and $\underline{\mathbf{1}}$ be the constant sequence 1. Then an arbitrary $\mathbf{x} = \{x_n\} \in c$ can always be expressed as $\mathbf{x} = x_0\underline{\mathbf{1}} + \sum_{n=1}^{\infty}(x_n - x_0)e^{(n)} = x_0\mathbf{1} + \lim_k \sum_{n=1}^{k}(x_n - x_0)e^{(n)}$ where $x_0 = \lim x_n$. Here the limit converges in the topology of c because $\lim |x_n - x_0| = 0$. Any bounded linear functional $\phi \in c^*$ satisfies

$$\phi(\mathbf{x}) = x_0\phi(\underline{\mathbf{1}} + \lim_k \sum_{n=1}^{k} (x_n - x_0)\phi(e^{(n)}) = x_0\phi(\underline{\mathbf{1}} + \lim_k \sum_{n=1}^{k} (x_n - x_0)\phi_n$$

where $\phi_n = \phi(e^{(n)})$. Since $c_0 \subseteq c, \phi|_{c_0}$ is a bounded linear functional. Therefore $\{\phi_n\}_{n=1}^{\infty} \in \ell_1$ and we can legitimately rearrange terms to write

$$\phi(\mathbf{x}) = x_0(\phi(\underline{\mathbf{1}}) - \sum_{n=1}^{\infty} \phi_n) + \sum x_n\phi_n = x_0\phi_0 + \sum x_n\phi_n,$$

where $\phi_0 = \phi(\underline{\mathbf{1}}) - \sum_{n=1}^{\infty} \phi_n$. Therefore $|\phi(\mathbf{x})| \leqslant \|\mathbf{x}\|(|\phi_0| + \sum_{n=1}^{\infty} |\phi_n|)$.

Claim: $\|\phi\| = \sum_{n=0}^{\infty} |\phi_n|$.

Proof of claim. We denote by sgn the signum function given by $\mathrm{sgn}(x) = 1$ for $x \geqslant 0$, $\mathrm{sgn}(x) = -1$ for $x < 0$. For each N let $x^{(N)} \in B_{c_0}$, the unit ball of c_0, be the sequence

$$x_n^{(N)} = \begin{cases} \mathrm{sgn}(\phi_n) \text{ if } 1 \leqslant n \leqslant N \\ \mathrm{sgn}\phi_0 \text{ otherwise} \end{cases} .$$

Then for all N we have

$$|\sum_{n=0}^{N} |\phi_n| + (\mathrm{sgn}\phi_0) \sum_{n=N+1}^{\infty} \phi_n| = |\phi(x^{(N)}| \leqslant \|\phi\| \|x^{(N)}\| = \|\phi\|.$$

Since this happens for all N and $\lim_N \sum_{n=N+1}^{\infty} \phi_n = 0$, we get $\sum_{n=0}^{\infty} |\phi_n| \leqslant \|\phi\|$. We already have shown $\|\phi\| \leqslant \sum_{n=0}^{\infty} |\phi_n|$. Therefore $\|\phi\| = \sum |\phi_n|$. □

We have established a linear isometry $\Phi : c_0^* \ni \phi \mapsto \{\phi_n\} \in \ell_1$. Only thing that remains to be shown is this is onto. But that is obvious because given any $\{\phi_n\} \in \ell_1$ we can define $\phi \in c_0^*$ as $\phi(\mathbf{x}) = \sum_n x_n \phi_n$. This series converges because $\{x_n\}$ is bounded and $\sum |\phi_n| < \infty$. Clearly $\Phi(\phi) = \{\phi_n\}$. □

Proposition 2.4.3. The spaces c and c_0 cannot be linearly isometrically isomorphic.

Proof. Let a be a real number less than $1/2$ in absolute value. Then $a = \frac{1}{2}((a + \frac{1}{8}) + (a - \frac{1}{8}))$. By doing this for each component for sufficiently large n we can conclude that every $\mathbf{x} \in c_0$ of norm less than or equal to one can always be expressed as $x = \frac{1}{2}(\mathbf{y} + \mathbf{z})$ with $\|\mathbf{y}\|, \|\mathbf{z}\| \leqslant 1, \mathbf{y}, \mathbf{z} \in c_0$ different from $\mathbf{x}$. But 1 cannot be written as $\frac{1}{2}(a + b)$ with $a \neq 1 \neq b, |a|, |b| \leqslant 1$. Therefore $\underline{\mathbf{1}}$ cannot be expressed as $\frac{1}{2}(\mathbf{y} + \mathbf{z})$ with $\|\mathbf{y}\|, \|\mathbf{z}\| \leqslant 1, \mathbf{y}, \mathbf{z} \in c$ different from $\underline{\mathbf{1}}$. Now in case we had a linear isometric isomorphism $T : c_0 \to c$ then from a decomposition of $T^{-1}(\underline{\mathbf{1}})$ we would have obtained one for $\underline{\mathbf{1}}$! □

Week 3

Geometric Formulation/Meaning of Hahn-Banach Theorems

3.1 The Minkowski functional

Definition 3.1.1. Let E be a vector space and $A \subseteq E$ a subset. We say A is absorbing if every $x \in E$ lies in some $t \cdot A$ for some $t = t(x) > 0$. Note that an absorbing set always contains zero. We say that A is balanced if $x \in A, |\lambda| \leqslant 1$ implies $\lambda x \in A$.

Definition 3.1.2. The Minkowski functional of an absobing set A is defined by

$$p_A = \inf\{t > 0 : t^{-1}x \in A\}.$$

Theorem 3.1.3. *Let p be a seminorm on a vector space E. Then $A = \{x : p(x) < 1\}$ is a convex, balanced, absorbing set and $p = p_A$.*

Proof. Only thing we need to verify is $p = p_A$. If $x \in E$ and $s > p(x)$ then $s^{-1}x \in A$. Therefore $p_A(x) \leqslant p(x)$. On the other hand if $0 < t \leqslant p(x)$, then $t^{-1}x \notin A$. Hence $p(x) \leqslant p_A(x)$. □

Definition 3.1.4. A vector space E endowed with a topology is called a topological vector space if it is Hausdorff and the operations of addition and scalar multiplication are continuous.

Theorem 3.1.5. *Let A be a convex absorbing subset of a vector space E and p_A its Minkowski functional. Then*

1. *p_A is subadditive, i.e., $p_A(x+y) \leqslant p_A(x) + p_A(y), \forall x, y \in E$.*
2. *p_A is positively homogeneous.*
3. *If A is balanced then p_A is a seminorm.*

4. *If* E *is a topological vector space and* A *is open then* $A = \{x \in E : p_A(x) < 1\}$.

Proof. (1)For all $\epsilon > 0$ we have λ, μ such that $p_A(x) \leqslant \lambda < p_A(x) + \epsilon, p_A(y) \leqslant \mu < p_A(y) + \epsilon$ and $\frac{x}{\lambda}, \frac{y}{\mu} \in A$. The convexity of A implies

$$\frac{x+y}{\lambda+\mu} = \frac{\lambda}{\lambda+\mu}\frac{x}{\lambda} + \frac{\mu}{\lambda+\mu}\frac{y}{\mu} \in A.$$

Therefore $p_A(x+y) \leqslant \lambda + \mu < p_A x) + p_A(y) + 2\epsilon$. Since ϵ is arbitrarily small, we obtain subadditivity.

(2), (3) Easily follows from the definition.

(4) Let $x \in A$. There exists an open neighborhood V of origin such that $x+V \subseteq A$. Since scalar multiplication is continuous there exists $\epsilon > 0$ such that $\epsilon.x \in V$. Then $(1+\epsilon)x \in A$. Therefore $p_A(x) \leqslant (1+\epsilon)^{-1} < 1$. Conversely suppose that $x \in E$ satisfies $p_A(x) < 1$. Then there exists $\epsilon \geqslant 0$ such that $\frac{x}{p(x)+\epsilon} \in A$ and $p_A(x) + \epsilon < 1$. Exploiting the convexity of A we get $x = (p(x)+\epsilon)\frac{x}{p(x)+\epsilon} + (1-p(x)-\epsilon).0 \in A$. □

Theorem 3.1.6. *Let* E *be a topological vector space over* $\mathbb{R}$ *and* A *be a convex open neighborhood of the origin. Let* $x_0 \notin A$*, then there is a hyperplane separating* x_0 *from* A*, in other words there is a continuous linear functional* $\ell \in E^*$ *such that*

$$\ell(x_0) = 1 \textit{ and } \ell(x) < 1, \quad \forall x \in A.$$

Proof. In a TVS scalar multiplication is continuous and A contains the origin. Therefore given any $x \in E$, the sequence x/n converges to 0, hence eventually enters the open neighborhood A. This shows that A is absorbing. Let p_A be the Minkowski functional of A. Then by theorem (3.1.5) we know that p_A is subadditive, positively homogeneous and $A = \{x \in E : p_A(x) < 1\}$. Since $x_0 \notin A$, we have $p_A(x_0) \geqslant 1$. On the one dimensional space spanned by x_0 define $\ell(\lambda x_0) = \lambda$. Then for $\lambda > 0$, $\ell(\lambda x) = \lambda \leqslant p_A(\lambda x_0)$. If $\lambda \leqslant 0$, then $\ell(\lambda x_0) = \lambda \leqslant 0 \leqslant p_A(\lambda x_0)$. In any case for any x from the subspace spanned by x_0 we have $\ell(x) \leqslant p_A(x)$. By theorem (1.1.2) we can extend ℓ to a linear map denoted by the same symbol ℓ on E such that $\ell(x) \leqslant p_A(x), \forall x \in E$. Then ℓ is continuous because if $x \in (-A) \cap A$, then $-1 < \ell(x) < 1$. □

Theorem 3.1.7. *Suppose* A *and* B *are disjoint nonempty convex sets in a topological vector space* E*. If* A *is open there exists* $\phi \in E^*$ *and* $\gamma \in \mathbb{R}$ *such that*

$$\Re\phi(x) < \gamma \leqslant \Re\phi(y), \forall x \in A, \forall y \in B.$$

If the scalar field is $\mathbb{R}$ *then* $\Re\phi := \phi$.

Proof. We will first do the case where the scalar field is $\mathbb{R}$. Fix $a_0 \in A$ and $b_0 \in B$. Put $x_0 = b_0 - a_0$ and $C = A - B + x_0$. Then C is open because it is a union of open sets $A - b + x_0, b \in B$. Clearly C is convex and contains the origin. Also $x_0 \in C$, because A and B are disjoint. Using theorem (3.1.6) obtain a continuous linear functional ϕ

such that $\phi(x_0) = 1$ and $\phi(x) < 1, \forall x \in C$. If $a \in A, b \in B$, then $\phi(a - b + x_0) = \phi(a) - \phi(b) + 1 < 1$. Therefore, $\phi(a) < \phi(b)$. Let $\gamma = \inf\{\phi(b) : b \in B\}$. Then $\phi(a) \leqslant \gamma, \forall a \in A$. Since A is open we must have $\phi(a) < \gamma, \forall a \in A$.

If the scalar field is $\mathbb{C}$, there is a continuous real linear map ϕ_1 satisfies the assertion. If ϕ is the associated complex linear map whose real part is ϕ_1, then $\phi \in E^*$ and does the job. □

Definition 3.1.8. A topological vector space is said to be locally convex if every point or equivalently origin has a neighborhood basis consisting of convex open sets.

Corollary 3.1.9. Let B be a closed and convex subset of a locally convex space E and $x_0 \notin B$ then there exists $\phi \in E^*$ such that $\phi(x_0) < \inf\{\phi(x) : x \in B\}$.

Proof. Let A be a convex neighborhood of x_0 disjoint from B. Now apply theorem (3.1.7) □

Lemma 3.1.10 (Topological lemma)**.** *Let E be a topological vector space, $C \subseteq E$ be a compact set and $D \subseteq E$ be a closed set. Then $C + D$ is closed.*

Proof. Since you are familiar with nets we will prove this using nets. Let $\{x_\alpha + y_\alpha\}_{\alpha \in A} \subseteq C+D$ be a convergent net with $\lim_\alpha (x_\alpha + y_\alpha) = z$. Since C is compact there exists a subnet $\{x_\beta\}$ converging to some $x \in C$. Then $\lim_\beta y_\beta = \lim_\beta (x_\beta + y_\beta - x_\beta) = z - x \in D$. So, we have $z = x + y \in C + D$. □

Theorem 3.1.11. *Let E be a locally convex space. Suppose $A, B \subseteq E$ are convex sets with A compact and B closed, $A \cap B = \emptyset$. Then there exists a linear continuous map $\phi : E \to \mathbb{K}$ and $\alpha, \beta \in \mathbb{R}$ such that*

$$\Re\phi(x) \leqslant \alpha < \beta \leqslant \Re\phi(y), \forall x \in A, \forall y \in B.$$

Proof. Consider the convex set $C = B - A$. By the topological lemma C is closed and $0 \notin C$, because $A \cap B = \emptyset$. Since E is locally convex there exists a convex open $D \subseteq E \setminus C$ containing the origin. In particular $C \cap D = \emptyset$. By theorem (3.1.7) we get a continuous linear map $\phi \in E^*$ and $\gamma \in \mathbb{R}$ such that

$$\Re\phi(d) < \gamma \leqslant \Re\phi(c), \forall d \in D, \forall c \in C.$$

Since $0 \in D, \gamma > 0$. The inequality $\Re\phi(c) \geqslant \gamma, \forall c \in C$ gives $\Re\phi(b) - \Re\phi(a) \geqslant \gamma > 0, \forall b \in B, \forall a \in A$. Let $\beta = \inf_{b \in B} \Re\phi(b), \alpha = \sup_{a \in A} \Re\phi(a)$. Then $\beta \geqslant \alpha + \gamma$ and we are done. □

3.2 Markov-Kakutani fixed point theorem

As a cute application of theorem (3.1.11) we discuss a proof of Markov-Kakutani fixed point theorem for locally convex spaces due to Dirk Werner.

Theorem 3.2.1 (Markov-Kakutani fixed point theorem)**.** *Let C be a compact convex set in a locally convex space E. A continuous map $T : C \to C$ is said to be affine if* $T(\lambda x + (1-\lambda)y) = \lambda T(x) + (1-\lambda)T(y), \forall x, y \in C, \forall \lambda \in [0,1]$. *Every commuting family* $\{T_i\}_{i \in I}$ *of continuous affine endomorphisms of C has a common fixed point.*

Lemma 3.2.2. *Let C be a compact convex set in a locally convex Hausdorff space E and let $T : C \to C$ be a continuous affine transformation. Then T has a fixed point.*

Proof. Let $\Delta = \{(x,x) : x \in C\}$ be the diagonal in C and $\Gamma = \{(x, Tx) : x \in C\}$. If T has no fixed point then $\Delta \cap \Gamma = \emptyset$. Both Δ and Γ are compact convex sets in $E \times E$. By the Hahn-Banach theorem (3.1.11) we get continuous linear functionals ϕ_1, ϕ_2 and $\alpha, \beta \in \mathbb{R}, \alpha < \beta$ such that

$$\Re(\phi_1(x) + \phi_2(x)) \leqslant \alpha < \beta \leqslant \Re(\phi_1(y) + \phi_2(Ty)).$$

Consequently $\Re(\phi_2(Tx) - \phi_2(x)) \geqslant \beta - \alpha > 0$. Iterating this inequality we get $\Re(\phi_2(T^n x) - \phi_2(x)) \geqslant n(\beta - \alpha) \to \infty$ for arbitrary $x \in C$. This makes the sequence $\{\Re\phi_2(T^n(x))\}_n$ unbounded contradicting the compactness of $\Re\phi_2(C)$. □

Proof of Markov-Kakutani fixed point theorem. Let C_i be the fixed points of T_i. Then $C_i \neq \emptyset, C_i$ is compact and convex. We need to show $\cap C_i \neq \emptyset$. But that will follow once we establish finite intersection property. Since $T_i T_j = T_j T_i, T_i(C_j) \subseteq C_j$. Hence $T_i|_{C_j}$ has a fixed point by lemma. In other words $C_i \cap C_j \neq \emptyset$. An obvious induction shows $\cap_{i \in F} C_i \neq \emptyset, \forall$ finite $F \subseteq I$. □

3.3 Weak topology

Definition 3.3.1 (Weak topology determined by a family)**.** Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of topological spaces and X a set. Suppose we have a collection of functions $\{f_\alpha : X \to X_\alpha\}_{\alpha \in A}$. Then the weak topology determined by this data is the weakest topology on X that makes all the f_α's continuous.

Remark 3.3.2. The collection $\{\cap_{i=1}^n f_{\alpha_i}^{-1}(U_{\alpha_i}|n \in \mathbb{N}, \alpha_1 \ldots, \alpha_n \in A, U_{\alpha_i} \subseteq X_{\alpha_i}$ is open $\forall i\}$ is a basis for the weak topology on X.We have the following characterisation of net convergence in weak topology. A net $\{x_\beta\}$ converges in weak topology to $x \in X$ iff $\forall \alpha \in A, \lim_\beta f_\alpha(x_\beta) = f_\alpha(x)$. A function $f : Y \to X$ from another topological space Y to X is continuous iff $f_\alpha \circ f : Y \to f_\alpha$ is continuous for all $\alpha \in A$.

Example 3.3.3 (Product topology:)**.** Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of topological spaces and

$$X = \prod X_\alpha := \{x : A \to \cup_{\alpha\in A} X_\alpha | \forall \alpha \in A, x(\alpha) \in X_\alpha\}.$$

This set exists by axiom of choice. Let $\pi_\alpha : X \to X_\alpha$ be the map $\pi_\alpha : x \mapsto x(\alpha)$. Then the weak topology determined by the family $\{\pi_\alpha : X \to X_\alpha\}_{\alpha\in A}$ is called the product topology on $\prod X_\alpha$. A net $\{x_\gamma\}_{\gamma\in\Gamma} \subseteq \prod X_\alpha$ converges to x iff $x_\gamma(\alpha) \to x(\alpha), \forall \alpha \in A$.

Exercise 3.3.4 (Characterization of products by universal properties)**.** The product topological space X satisfies two properties.

1. Suppose Y is a topological space and for each $\alpha \in A$ we have a continuous function $g_\alpha : Y \to X_\alpha$. Then there is a continuous function $g : Y \to X$ such that $\pi_\alpha \circ g = g_\alpha, \forall \alpha$.
2. Let Z be a topological space so that for each $\alpha \in A$ we have continuous maps $p_\alpha : Z \to X_\alpha$ and whenever we have continuous maps $g_\alpha : Y \to X_\alpha, \forall \alpha$, there exists unique $g : Y \to Z$ satisfying $p_\alpha \circ g = g_\alpha, \forall \alpha \in A$. Then Z is homeomorphic with X. In other words this property characterises product topology.

Definition 3.3.5. Let E be a $\mathbb{K}$-vector space and $A \subseteq L(E;\mathbb{K})$ a collection of linear maps. Then the weak topology on E determined by this family is denoted by $\sigma(E;A)$. So whenever we talk about the space $\sigma(E;A)$ we mean E endowed with the weak topology determined by A. In particular if E is a topological vector space and E^* is the collection of continuous linear functionals on E then $\sigma(E;E^*)$ is called the weak topology on E. Also each $x \in E$ determines a linear map ϕ_x on E^* given by $\phi_x : x^* \mapsto x^*(x)$. Then $\sigma(E^*;\{\phi_x : x \in E\})$ is called the weak* topology on E^*.

Remark 3.3.6. If E is a normed linear space then $\sigma(E^*; j_E(E))$ is E^* with weak* topology.

Theorem 3.3.7 (Mazur)**.** *Let E be a locally convex space and K a convex subset of E. Then K is weakly closed if and only if it is closed.*

Proof. An arbitrary subset of E is closed provided it is weakly closed. So, we only need to show that K is weakly closed assuming it is closed. If possible let x_0 be a point in the weak closure of K which is not in K. Then there is a net $x_\alpha \in K$ such that $\forall \psi \in E^*, \psi(x_\alpha) \to \psi(x_0)$. At the same time by corollary (3.1.9) there exists $\phi \in E^*$ such that $\phi(x_0) < \inf\{\phi(x) : x \in K\}$. Clearly these two contradict each other because $\{\phi(x_\alpha)\}$ can not converge to $\phi(x_0)$. □

Theorem 3.3.8 (Banach-Alaoglu Theorem)**.** *Let E be a Banach space, then B_{E^*} the closed unit ball in E^* is weak* compact.*

Proof. For $x \in E$ let $B_x = \{z \in \mathbb{C} : |z| \leqslant \|x\|\}$. Then by Tychonoff's theorem $B = \prod_{x\in E} B_x$ is compact. It suffices to show that the unit ball in E^* can be embedded in B. Let $\phi : B_{E^*} \to B$ be given by $\phi(x^*) = (< x^*, x >)$. Then clearly ϕ is one to one. To show B_{E^*} is compact it suffices to show that ϕ^{-1} is continuous and $\phi(B_{E^*})$ is closed. ϕ^{-1} is continuous because by definition of weak* topology if $\phi(x^*_\alpha)$ is a net converging in weak* topology to x^*, then the x-th coordinate of $\phi(x^*_\alpha)$ is converging to the x-th coordinate of $\phi(x)$, i.e., $\phi(x^*_\alpha)$ is converging to $\phi(x^*)$. Clearly ϕ is one to one. □

Corollary 3.3.9. Let E be a reflexive Banach space. Then the unit ball of E is weakly compact.

Proof. By reflexivity the weak topology of E coincides with the weak* topology of E^{**}. □

Remark 3.3.10. The converse is also true but we are not proving that now. One way to see it is through Goldstine'e theorem.

3.4 Stone-Weirstrass Theorem

Theorem 3.4.1 (Weirstrass Theorem)**.** *Polynomials are dense in* $C[a, b]$.

Proof. Enough to prove for the interval $[0, 1]$. Let $\Omega = \{0, 1\}$ and $\mathcal{P}_\Omega$ be the power set of Ω. Consider the probability space $(\Omega, \mathcal{P}_\Omega, P_p)$ where $P_p(\{1\}) = p, P_p(\{0\}) = 1 - p$. Let $(\Omega_n, \mathfrak{S}_n, P_{n,p})$ be the n-fold product of $(\Omega, \mathcal{P}_\Omega, P_p)$. Then $\Omega_n = \Omega^n$, the n-fold cartesian product of Ω. Consider the random variables $X_j : (\Omega_n, \mathfrak{S}_n) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$, given by $X_j : \omega \mapsto \omega_j$ where $\omega = (\omega_1, \dots, \omega_n) \in \Omega^n$. Note that for all $\eta_1, \dots, \eta_n \in \Omega$,

$$P_{n,p}(\{\omega : X_j(\omega) = \eta_j, 1 \leqslant j \leqslant n\}) = \prod_{j=1}^{n} P_{n,p}(\{\omega : X_j(\omega) = \eta_j\}) = p^{\sum \eta_j}(1-p)^{n-\sum \eta_j}.$$

If we denote $\frac{1}{n}\sum_{j=1}^{n} X_j$ by $\overline{X}_n$ then

$$\mathbb{E}_p\overline{X}_n := \int_{\Omega_n} \overline{X}_n dP_{n,p} = \sum_{\omega\in\Omega_n} \overline{X}_n(\omega)P_{n,p}(\{\omega\}) = p$$

and

$$\mathbb{V}ar_p(\overline{X}_n) := \int_{\Omega_n} (\overline{X}_n - \mathbb{E}_p\overline{X}_n)^2 dP_{n,p} = \sum_{\omega\in\Omega_n} (\overline{X}_n(\omega) - \mathbb{E}_p\overline{X}_n)^2 P_{n,p}(\{\omega\}) = \frac{p(1-p)}{n}.$$

By Chebychev's inequality

$$P_{n,p}(\{\omega : |\overline{X}_n(\omega) - \mathbb{E}_p\overline{X}_n| > \delta\}) \leqslant \frac{\mathbb{V}ar_p(\overline{X}_n)}{\delta^2} = \frac{p(1-p)}{n\delta^2} \leqslant \frac{1}{4n\delta^2}.$$

With this background we are now ready to approximate f by polynomials. Idea of the proof is as follows: $\overline{X}_n$ is close to p with high probability. Therefore we expect that $\mathbb{E}_p f(\overline{X}_n)$ should be close to $f(p)$. So, let us try to estimate the difference $|\mathbb{E}_p f(\overline{X}_n) - f(p)|$ or equivalently $|\mathbb{E}_p(f(\overline{X}_n) - f(p))|$. Since $f \in C[0,1]$, it is uniformly continuous. Therefore given $\epsilon > 0$ there is $\delta > 0$ such that

$$|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \epsilon/2.$$

Let $N \in \mathbb{N}$ be such that $\frac{\|f\|}{2\epsilon\delta^2} < N$. Then for $n \geqslant N$,

$$\begin{aligned}
|\mathbb{E}_p(f(\overline{X}_n) - f(p))| &= |\int (f(\overline{X}_n) - f(p)) dP_{n,p}| \\
&\leqslant \int |(f(\overline{X}_n) - f(p))| dP_{n,p} \\
&= \int |(f(\overline{X}_n) - f(p))| I(|\overline{X}_n - p| < \delta) dP_{n,p} + \\
&\quad \int |(f(\overline{X}_n) - f(p))| I(|\overline{X}_n - p| \geqslant \delta) dP_{n,p} \\
&< \frac{\epsilon}{2} + \frac{2\|f\| \mathbb{V}ar_p(\overline{X}_n))}{\delta^2} \\
&< \frac{\epsilon}{2} + \frac{\|f\|}{2n\delta^2} < \epsilon
\end{aligned}$$

We are done once we note that

$$B_n(f)(p) = \mathbb{E}_p f(\overline{X}_n) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) p^k (1-p)^{n-k}$$

is a polynomial in p and we have proved

$$\|B_n(f) - f\| < \epsilon.$$

□

Corollary 3.4.2. Let X be a compact topological space and $A \subseteq C(X, \mathbb{R})$ a closed unital subalgebra, then

- if $f \in A$, then $|f| \in A$.
- A is a lattice, that is $f, g \in A \Longrightarrow f \wedge g, f \vee g \in A$.

Proof. (i) There is a sequence of polynomials $p_n(t)$ such that on the interval $[-\|f\|, \|f\|]$ $p_n(\cdot)$ uniformly converges to $t \mapsto |t|$. Clearly $p_n(f) \in A$ and uniformly converges to $|f|$. Since A is closed $|f| \in A$.

(ii) For this just note that

$$\begin{aligned}
f \wedge g &= \frac{1}{2}(f + g - |f - g|) \\
f \vee g &= \frac{1}{2}(f + g + |f - g|)
\end{aligned}$$

□

Theorem 3.4.3 (Stone-Weirstrass Theorem)**.** *Let X be a compact topological space and A a closed unital subalgebra of $C(X, \mathbb{R})$ which separates points. Then $A = C(X, \mathbb{R})$.*

Proof. Let $f \in C(X, \mathbb{R})$. We will show that $f \in A$. Let $s, t \in X$. Since A separates points there is $g \in A$ such that $g(s) \neq g(t)$. For some $\lambda, \mu \in \mathbb{R}$ let

$$\widetilde{g}(\nu) = \mu + (\lambda - \mu)\frac{g(\nu) - g(t)}{g(s) - g(t)}.$$

Then $\widetilde{g} \in A$ and $\widetilde{g}(s) = \lambda, \widetilde{g}(t) = \mu$. Thus for $s \neq t$, there exists $f_{s,t} \in A$ such that

$$\begin{aligned} f_{s,t}(s) &= f(s) \\ f_{s,t}(t) &= f(t). \end{aligned}$$

Let $U_t = \{\nu \in X : f_{s,t}(\nu) < f(\nu) + \epsilon\}$. Then $t \in U_t$ and U_t is open. Since X is compact there is $t_1, \cdots, t_n$ such that $X = \cup_i U_{t_i}$. Define

$$g_s = \min_{1 \leqslant i \leqslant n} f_{s,t_i}.$$

Then,

$$\begin{aligned} g_s &\in A \\ g_s(s) &= f(s) \\ g_s &< f + \epsilon. \end{aligned}$$

Now define

$$V_s = \{\nu \in X : g_s(\nu) > f(\nu) - \epsilon\}.$$

Note that V_s is open and $s \in V$. Thus $X = \cup_{s \in X} V_s$ and there is $s_1, \cdots, s_m$ such that $X = \cup_{i+1}^m V_{s_i}$. Put

$$g = \max_{1 \leqslant i \leqslant m} g_{s_i}.$$

Then $g \in A$ and

$$f - \epsilon < g < f + \epsilon.$$

That is $\|f - g\} < \epsilon$. Since A is closed $f \in A$. □

Week 4

Baire Category Theorem and its Consequences

This week we discuss the other basic tools of functional analysis.

4.1 Baire Category Theorem

Theorem 4.1.1 (Baire Category Theorem)**.** *Let X be a complete metric space. If* U_n *is a sequence of open dense sets in* X *then* $\cap U_n$ *is also dense in* X.

Proof. Let d be a distance defining the topology of X. Let B be an open ball and we want to show that $B \cap U_n \neq \phi$. Clearly it suffices to show that for any closed ball $\bar{B} \cap U_n \neq \phi$. Replacing X by $\bar{B}$ it suffices to show that $\cap U_n \neq \phi$. We shall define a sequence x_n and positive real numbers r_n such that (i) $B'(x_n, r_n) \subseteq U_n \cap B(x_{n-1}, r_{n-1})$ and (ii) $r_n < 1/n$. Here $B'(u, r)$ denotes the closed ball with center u and radius r. Start with $x_1 \in U_1$ and $r_1 < 1$ such that $B'(x_1, r_1) \subseteq U_1$. After defining $X_1, \cdots, x_{n-1}$ choose $x_n \in U_n \cap B(x_{n-1}, r_{n-1})$ and $r_n < 1/n$ such that (ii) holds. One can do this because U_n is dense and $U_n \cap B(x_{n-1}, r_{n-1})$ is open. Clearly $d(x_n, x_{n+p}) < r_n < 1/n$ for each $n \geqslant 1$ and p. Hence x_n is a Cauchy sequence and by hypothesis it converges to some $x \in E$. Since $x_{n+p} \in B'(x_n, r_n)$ for all $p > 1$, $x \in B'(x_n, r_n) \subseteq U_n$ for each n. Therefore $x \in \cap U_n$. □

Corollary 4.1.2. Let X be a complete metric space and C_n a sequence of closed sets such that $X = \cup C_n$. Then at least one of them has nonempty interior.

Proof. On the contrary suppose every C_n has empty interior. Let $U_n = X \setminus C_n$, then U_n's are dense open subsets of X and by Baire's theorem $\cap U_n$ is dense. On the other hand

$$\cap U_n = \cap(X \setminus C_n) = X \setminus (\cup C_n) = X \setminus X = \phi$$

a contradiction. □

4.2 The uniform boundedness principle and an application

Theorem 4.2.1 (Uniform Boundedness Principle). *Let* $\{T_\alpha : E \to F\}_{\alpha \in A}$ *be a family of continuous linear maps such that for each* $x \in E$ *there exists* M_x *such that* $\sup_\alpha \|T_\alpha(x)\| \leqslant M_x\|x\|$*, then there exists* M *such that* $\sup_\alpha \|T_\alpha\| \leqslant M$.

Proof. Let $C_n = \{x \in E : \forall \alpha, \quad \|T_\alpha(x)\| \leqslant n\|x\|\}$. Then clearly each C_n is closed and they cover E. Therefore at least one of them say C_k contains a ball of radius r around x_0 for some r and x_0. Hence $\|T_\alpha(x\| \leqslant k\|x\|$ whenever $\|x - x_0\| < r$ and consequently for x with $\|x - x_0\| \leqslant r$ using $\|x\| \leqslant \|x_0\| + r$ we get

$$\|T_\alpha(x - x_0)\| \leqslant \|T_\alpha(x)\| + \|T_\alpha(x_0)\| \leqslant k\|x\| + k\|x_0\| \leqslant k(2\|x_0\| + r).$$

Therefore $\sup_\alpha \|T_\alpha\| \leqslant \frac{k(2\|x_0\|+r)}{r}$. □

Corollary 4.2.2. Let E be a Banach space. Let X be a weakly bounded subset of E. That means for all $\phi \in E^*$, $\phi(X)$ is a bounded subset of $\mathbb{K}$. Then X is a norm bounded subset of E.

Proof. Let $j : E \to E^{**}$ be the canonical embedding. Then by hypothesis

$$\forall \phi \in E^*, \exists M_\phi \text{ such tat } \sup_{x \in X} \|j(x)(\phi)\| < M_\phi.$$

By the uniform boundedness principle there exists M such that

$$\sup_{x \in X} \|x\| = \sup_{x \in X} \|j(x)\| < M.$$

□

4.3 A typical application

Let $1 < p < \infty$ and $\{\alpha_n\}$ be a sequence of scalars such that $\sum \alpha_n \beta_n$ converges for all $\{\beta_n\} \in \ell_p$. Then $\{\alpha_n\} \in \ell_q$. To see this consider the linear functional $T_N \in \ell_p^*$ given by $T_N(\{\beta_n\}) = \sum_{n=1}^N \alpha_n \beta_n$. From convergence of $\sum \alpha_n \beta_n$ we conclude that the hypothesis of UBP is met. Therefore UBP gives us M such that $M > \sup_N \|T_N\| = \sup_N \sqrt[q]{\sum_{n=1}^N |\alpha_n|^q}$. Therefore $\sum_{n=1}^\infty |\alpha_n|^q \leqslant M < \infty$.

4.4 Quotient spaces

Now that we have some idea about bounded linear maps on normed linear spaces we can ask how about extending some of the results of linear algebra to normed linear spaces. The first theorem we learnt was the first isomorphism theorem. Recall that first isomorphism theorem says if T is a linear map from a linear space E onto another linear space F then T induces an isomorphism $q_T : E/\ker T \to F$. Now if we want to extend this to normed linear spaces first thing we need is the notion of quotients.

Definition/Proposition 4.4.1. Let E be a normed linear space and $F \subseteq E$ a closed subspace. Then $\|[x]\| := \inf\{\|x+y\| : y \in F\}$ defines a norm on the vector space E/F.

Proof. Let $x_1, x_2 \in E$. Then $\forall y_1, y_2 \in F$ we have

$$\|x_1 + y_1 + x_2 + y_2\| \leqslant \|x_1 + y_1\| + \|x_2 + y_2\|.$$

Taking infimum over both sides as y_1, y_2 varies over F we get $\|[x_1 + x_2]\| \leqslant \|[x_1]\| + \|[x_2]\|$. Similarly we get $\|[\lambda x]\| = |\lambda|\|[x]\|$. Finally note that $\|[x]\| = 0$ iff $x = \lim y_n$ for some sequence $\{y_n\} \subseteq F$. Since F is closed, this happens iff $x \in F$. In other words $[x] = 0 \in E/F$. □

Lemma 4.4.2. *Let E be a normed linear space. Then E is complete iff convergence of $\sum \|x_n\|$ implies convergence of $\sum x_n$.*

Proof. Only if part is easy and we only show the if part. Let $\{x_n\}$ be a Cauchy sequence in E. Then we can extract a subsequence $\{x_{n_k}\}$ such that $\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k}, \forall k$. Then the series $\sum \|x_{n_{k+1}} - x_{n_k}\|$ converges. By our hypothesis $\sum_{k=1}^{N}(x_{n_{k+1}} - x_{n_k})$ converges. That means $x_{n_k} - x_{n_1}$ converges. In other words the subsequence $\{x_{n_k}\}$ converges. Since the original sequence is Cauchy from the convergence of a subsequence we conclude convergence of the whole sequence. □

Proposition 4.4.3. Let E be a Banach space and $F \subseteq E$ is a closed subspace. Then E/F with the quotient norm is a Banach space.

Proof. Let $\sum \|[x_n]\| < \infty$ to show completeness of E/F it is enough to show convergence of $\sum [x_n]$. For each n obtain $y_n \in F$ such that $\|x_n + y_n\| \leqslant \|[x_n]\| + \frac{1}{2^n}$. Then $\sum \|x_n + y_n\| < \infty$ and using completeness of E we conclude convergence of $\sum (x_n + y_n)$ say to x_0. In other words $\| \sum_{n=1}^{N}(x_n + y_n) - x_0\| \to 0$. Since $\sum_{n=1}^{N} y_n \in F$ we have

$$\| \sum_{n=1}^{N} [x_n] - [x_0]\| \leqslant \| \sum_{n=1}^{N} (x_n + y_n) - x_0\| \to 0.$$

Thus we have established $\lim \sum_{n=1}^{N} [x_n] = [x_0]$. □

4.5 An application of UBP to complex analysis

We briefly recall the basic concepts of Banach space valued holomorphic functions.

Definition 4.5.1. (1) Let Ω be an open subset of $\mathbb{C}$ and E a Banach space. A function $f : \Omega \to E$ is called weakly holomorphic if given any bounded linear functional ϕ on E, the function $\phi \circ f : \Omega \to \mathbb{C}$ is holomorphic.
(2) The function f is called strongly holomorphic if for all $z \in \Omega$

$$\text{Lim}_{w \to z} \frac{f(w) - f(z)}{w - z} \qquad \text{exists} .$$

Proposition 4.5.2. If $f : \Omega \to E$ is weakly holomorphic then f is norm continuous.

Proof. Suppose $0 \in \Omega$, and we will show that f is norm continous at zero. Let ϕ be a linear functional on E. Since $\phi \circ f$ is holomorphic,

$$\frac{\phi(f(z)) - \phi(f(0))}{z} = \frac{1}{2\pi i} \int_\Gamma \frac{\phi(f(w))}{(w - z)w} dw,$$

where Γ is the positively oriented circle of radius $2r$ for some small enough r and $0 < |z| < 2r$. For $|z| < r$ the right hand side is bounded by $r^{-1}C(\phi)$ for some constant $C(\phi)$ dependent on ϕ. So

$$|\phi\left(\frac{f(z) - f(0)}{z}\right)| \leqslant r^{-1}C(\phi) \qquad \text{for } 0 < |z| < r.$$

By the uniform boundedness principle, there is some constant c such that for $0 < z < r$, $\|\frac{f(z)-f(0)}{z}\| \leqslant c$. Therefore f is norm continuous at 0. □

4.6 Open mapping theorem and its main corollary

Theorem 4.6.1 (Open Mapping Theorem). *Let $T : E \to F$ be a continuous surjection, then T is an open mapping theorem.*

Lemma 4.6.2. *Let $T : E \to F$ be a bounded operator from a Banach space E to another Banach space F. Let B_E and B_F be the unit balls of E and F respectively. Suppose that $T(B_E)$ closure contains rB_F for some $r > 0$, then $T(B_E)$ contains rB_F.*

Proof. Let $y \in rB_F$ and $\delta \in (0, 1)$ such that $y' = \delta^{-1}y \in rB_F$. By the assumption, there exists $x_1 \in B_E$ such that $\|y' - T(x_1)\| < (1 - \delta)r$. Since $\overline{T((1 - \delta)B_E)}$ contains $(1 - \delta)rB_F$, there exists $x_2 \in (1 - \delta)B_E$ such that $\|y - T(x_1) - T(x_2)\| < r(1 - \delta)^2$. Since $\overline{T((1 - \delta)^2B_E)}$ contains $(1 - \delta)^2rB_F$, there exists $x_3 \in (1 - \delta)^2B_E$ such that $\|y - T(x_1) - T(x_2) - T(x_3)\| < r(1 - \delta)^3$. Continuing this process we get a sequence $x_n \in (1 - \delta)^{n-1}B_E$ such that

$$\|y - T(x_1) - T(x_2) - \cdots - T(x_n)\| < r(1 - \delta)^n.$$

Since $\sum \|x_n\|$ converges and E is complete the series $\sum x_n$ converges to x' say. Since T is continuous $T(x') = y'$ and $\|x'\| < \sum (1-\delta)^{-1} = \delta^{-1}$. Put $x = \delta x'$, then clearly $x \in B_E$ and $T(x) = \delta y' = y$. □

Open Mapping Theorem. We have to show that the image of an open ball around zero under T contains an open ball around zero. Since T is surjective, $F = \cup T(nB_E)$. But by the corollary to the Baire theorem we get closure of $T(mB_E)$ contains an open ball $V = y + \epsilon B_F$. Put $r = \frac{\epsilon}{2m}$ and take $z \in rB_F$. Since $y, y + 2mz \in V$, there exists sequences $y_n, {y'}_n \in T(mB_E)$ such that $\lim y_n = y, \lim {y'}_n = y + 2mz$. Hence $z_n = y_n - {y'}_n \in T(2mB_E)$ converges to $2mz$, and thus $\frac{1}{2m} z_n \in T(B_E)$ converges to z. Thus we can apply the previous lemma and conclude the proof. □

Remark 4.6.3 (A typical application). Let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on a linear space E turning E into a Banach space. Suppose there exists $C > 0$ such that $\|x\|_1 \leqslant C\|x\|_2, \forall x \in E$. Then there exists C' such that $\|x\|_2 \leqslant C'\|x\|_1, \forall x \in E$. To see this just observe that the identity map from $(E, \|\cdot\|_2)$ to $(E, \|\cdot\|_1)$ is a bijective continuous surjection. By the open mapping theorem this mapping has a continuous or equivalently bounded inverse. We can take C' to be the norm of the inverse.

Theorem 4.6.4 (Closed Graph Theorem). *Let E, F be Banach spaces and $T : E \to F$ a linear map such that the graph of T, $\Gamma = \{(x, T(x)) : x \in E\}$ is a closed subset of $E \times F$. Then T is continuous.*

Proof. The vector space $E \times F$ is a Banach space with the norm $\|(x, y)\| = \|x\|_E + \|y\|_F$. By hypothesis Γ is a closed subspace of a Banach space, hence Γ becomes a Banach space. Define $\pi_1 : \Gamma \to E$ as $\pi_1((x, T(x))) = x$ and $\pi_2 : E \times F \to F$, as $\pi_2((x, y)) = y$. By the open mapping theorem π_1^{-1} is a continuous linear map from X to Γ. But $T = \pi_2 \circ \pi_1^{-1}$, hence continuous. □

Proposition 4.6.5. Let $\|\cdot\|_{\mathcal{N}}$ be a norm on $C([0, 1])$ turning it into a Banach space. Also $\|f_n - f\|_{\mathcal{N}} \to 0$ implies $\lim f_n(x) = f(x), \forall x \in [0, 1]$. Then $\|\cdot\|_{\mathcal{N}}$ must be equivalent with the sup norm.

Proof. Because of remark (4.6.3) it is enough to show that the identity mapping from $(C([0, 1]), \|\cdot\|_{\sup})$ to $(C([0, 1]), \|\cdot\|_{\mathcal{N}})$ is continuous. We can appeal to closed graph theorem provided we show that the graph of identity mapping is closed. In other words if $\lim \|f_n - f\|_{\sup} = 0, \lim \|f_n - g\|_{\mathcal{N}} = 0$ then we must show $g = f$. But that follows from $g(x) = \lim f_n(x) = f(x)$. □

Week 5

Hilbert Spaces

We briefly introduce Hilbert spaces and classify them.

5.1 Hilbert Spaces

Definition 5.1.1. Let $\mathcal{H}$ be a vector space. A pre-inner product on $\mathcal{H}$ is a sesquilinear map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ such that

1. $\langle u, v \rangle = \overline{\langle v, u \rangle}, \forall u, v \in \mathcal{H}$.
2. $\langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle, \forall \alpha, \beta \in \mathbb{K}, \forall u, v \in \mathcal{H}$.
3. $\langle u, u \rangle \geqslant 0 \ \forall u \in \mathcal{H}$.

Definition 5.1.2. A **Pre-Hilbert Space** or a pre-inner product space is a pair consisting of vector space along with a pre-inner product.

Proposition 5.1.3 (Cauchy-Schwarz Inequality)**.** Let $\mathcal{H}$ be a vector space equipped with a pre-inner product, then

$$| < u, v > | \leqslant \sqrt{\langle u, u \rangle} \sqrt{\langle v, v \rangle}, \forall u, v \in \mathcal{H}.$$

Proof. Let $\langle u, v \rangle = re^{i\theta}, r \geqslant 0$. Note that if the scalar field is $\mathbb{R}$ then $\theta \in \{\pi, 0\}$. We will divide the proof in cases. The first one is$\langle u, u \rangle = \langle v, v \rangle = 0$.

$$\begin{aligned} 0 &\leqslant \langle u - e^{-i\theta} v, u - e^{-i\theta} v \rangle \\ &= \langle u, u \rangle + \langle v, v \rangle - e^{-i\theta} \langle u, v \rangle - e^{i\theta} \overline{\langle v, u \rangle} \\ &= -2r \leqslant 0. \end{aligned}$$

Thus we get $r = 0$ proving the inequality in this case. Next case is both $\langle u, u\rangle$ and $\langle v, v\rangle$ are not simultaneously zero. Without loss of generality we can assume that $\langle v, v\rangle \neq 0$. Let $t = -\frac{\langle u,v\rangle}{\sqrt{\langle v,v\rangle}}$, then,

$$\begin{aligned} 0 &\leqslant \langle u + tv, u + tv\rangle \\ &= \langle u, u\rangle + |t|^2\langle v, v\rangle - \frac{2|\langle u, v\rangle|^2}{\langle v, v\rangle} \\ &= \langle u, u\rangle + \frac{|\langle u, v\rangle|^2}{\langle v, v\rangle} - \frac{2|\langle u, v\rangle|^2}{\langle v, v\rangle} \\ &= \langle u, u\rangle - \frac{|\langle u, v\rangle|^2}{\langle v, v\rangle}. \end{aligned}$$

Now transferring $\frac{|\langle u,v\rangle|^2}{\langle v,v\rangle}$ to the other side and multiplying both sides by $\langle v, v\rangle$ we get the result. □

Corollary 5.1.4. We have $\langle u, v\rangle = 0$ whenever $\langle v, v\rangle = 0$.

Corollary 5.1.5. $N = \{v \in \mathcal{H} : \langle v, v\rangle = 0\}$ is a subspace.

Proof. Clearly N is closed under scalar multiplication. Only thing we need to show that it is closed under addition. Let $u, v \in N$. Then by the C-S inequality we get $\langle u, v\rangle = 0$. Thus $\langle u + v, u + v\rangle = 0$. □

Corollary 5.1.6. $\sqrt{\langle u, u\rangle} = \sup_{v:\langle v,v\rangle=1} |\langle u, v\rangle|$

Proof. If $\langle u, u\rangle = 0$ then both sides are zero. Otherwise by the C-S inequality left hand side is less than or equal to right hand side and taking $v = u/\sqrt{\langle u, u\rangle}$ we get the other inequality. □

Definition 5.1.7. Let $\mathcal{H}$ be a vector space. An inner product on $\mathcal{H}$ is a sesquilinear map $\langle\cdot, \cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ such that

1. $\langle\cdot, \cdot\rangle$ is a pre-inner product.
2. Positive definiteness: $\langle u, u\rangle = 0 \implies u = 0$.

An inner product space $(\mathcal{H}, \langle\cdot, \cdot\rangle)$ is a pair consisting of a vector space $\mathcal{H}$ along with an inner product on $\mathcal{H}$

Definition/Proposition 5.1.8. Let $(\mathcal{H}, \langle\cdot, \cdot\rangle)$ be an inner product space, then the map $\|\cdot\| : \mathcal{H} \to \mathbb{R}_+$ given by

$$\|v\| = \begin{cases} \sqrt{\langle v, v\rangle}, v \neq 0 \\ 0, \text{ for } v = 0. \end{cases}$$

is a norm on $\mathcal{H}$. This norm is referred as the norm associated with the inner product $\langle\cdot, \cdot\rangle$.

Proof. Let $u, v \in \mathcal{H}$. Only thing we need to verify is $\|u+v\| \leqslant \|u\| + \|v\|$. That follows from,

$$\begin{aligned} \|u+v\|^2 &= \langle u+v, u+v\rangle = \|u\|^2 + \|v\|^2 + 2\Re(\langle u, v\rangle) \\ &\leqslant \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| = (\|u\| + \|v\|)^2 \end{aligned}$$

□

Definition 5.1.9. An inner product space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is called a **Hilbert space** if $\mathcal{H}$ is complete with respect to the norm associated with the inner product.

Definition 5.1.10. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. A linear map $U : \mathcal{H}_1 \to \mathcal{H}_2$ is called **unitary** if it is one-to-one, onto and preserves inner products that is, $\langle Ux, Uy\rangle = \langle x, y\rangle$, for all $x, y \in \mathcal{H}_1$. The Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ are called unitarily equivalent if there is a unitary U from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Proposition 5.1.11. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces with dense subspaces S_1, S_2 respectively. Let $U : S_1 \to S_2$ be a bijection such that $\langle Ux, Uy\rangle = \langle x, y\rangle$, for all $x, y \in S_1$, then U extends to a unitary map denoted by the same symbol U from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Proof. Observe that $\|U(x)\| = \|x\|$, for all $x \in \mathcal{S}_1$. Therefore U converts Cauchy sequences to Cauchy sequences. If x is an element in $\mathcal{H}_1$ there is a sequence $\{x_n\}$ of elements of S_1 converging to x. Now $\{U(x_n)\}$ is also Cauchy and therefore converges to some limit. Define Ux as this limit. Clearly this is well defined. By playing the same game with U^{-1} we conclude that the extended map is bijective as well. Continuity of the innerproduct combined with the density of S_i's give $\langle Ux, Uy\rangle = \langle x, y\rangle$, for all $x, y \in \mathcal{H}_1$. □

Definition/Proposition 5.1.12. Let $(\mathcal{H}_{pre}, (\cdot,\cdot))$ be a pre-Hilbert space. Let $N = \{v \in \mathcal{H}_{pre} : (v, v) = 0\}$. Then $\langle u+N, v+N\rangle = (u, v)$ defines an inner product on $\mathcal{H}_{pre}/N$. Completion of $\mathcal{H}_{pre}/N$ with respect to the associated norm is called the Hilbert space associated with the pre-Hilbert space $\mathcal{H}_{pre}$.

Proof. By corollary (5.1.4) the sesquilinear form $\langle\cdot,\cdot\rangle$ is well defined. Only thing we need to verify is positive definiteness. Let $u \in \mathcal{H}_{pre}$ be such that $\langle u+N, u+N\rangle = (u, u) = 0$. Then $u \in N$ and consequently $u + N = N$. □

Proposition 5.1.13. Let $\mathcal{H}$ be a Hilbert space and $C \subseteq \mathcal{H}$ be a closed convex set. Then for all $x \notin C$ there exists unique $\tilde{z} \in C$ such that $\|x - \tilde{z}\| = \inf\{\|x - z\| : z \in C\}$. Verbally this means C has a unique point closest to x.

Proof. Uniqueness: Let $z_1, z_2 \in C$ be equidistant from x. In other words $\|x - z_1\| = \|x - z_2\|$. Then by the parallelogram identity

$$\|(x - z_1) + (x - z_2)\|^2 + \|(x - z_1) - (x - z_2)\|^2 = 2(\|x - z_1\|^2 + \|x - z_2\|^2)$$

Therefore
$$\|x - \frac{z_1 + z_2}{2}\|^2 + \frac{1}{4}\|z_1 - z_2\|^2 = \|x - z_1\|^2 = \|x - z_2\|^2.$$
So, either $z_1 = z_2$ or else their midpoint $\frac{z_1+z_2}{2}$ is a point from C closer to x.

Existence: Let $c = \inf\{\|x - z\|^2 : z \in C\}$. Then there exists a sequence $\{z_n\} \subseteq C$ such that $c \leqslant \|x - z_n\|^2 \leqslant c + \frac{1}{n}$. Then using parallelogram identity we get

$$\begin{aligned}\|z_n - z_m\|^2 &= 2(\|x - z_n\|^2 + \|x - z_m\|^2) - 4\|x - \frac{z_1 + z_2}{2}\|^2 \\ &\leqslant 2(c + 1/n + c + 1/m) - 4c = 2(1/n + 1/m).\end{aligned}$$

Since $\mathcal{H}$ is complete and C is closed $\{z_n\}$ converges to some $\tilde{z} \in C$. Using continuity of corm we conclude

$$\|x - \tilde{z}\| = \lim \|x - z_n\| = c = \inf\{\|x - z\|^2 : z \in C\}.$$ □

Proposition 5.1.14. Let $\mathcal{H}_0 \subseteq \mathcal{H}$ be a closed subspace and $x \notin \mathcal{H}_0$. Let $\tilde{z}$ be the unique solution to the minimization problem $\min\{\|x-z\| : z \in \mathcal{H}_0\}$. Then $\langle x-\tilde{z}, z\rangle = 0, \forall z \in \mathcal{H}_0$.

Proof. We do it for complex scalars. The real case is easier. Let $\lambda \in \mathbb{C}$ and $z \in \mathcal{H}_0$. Then
$$\|x - \tilde{z}\|^2 \leqslant \|x - \tilde{z} - \lambda z\|^2$$
So, for all such λ and z

$$-2\Re\langle x - \tilde{z}, \lambda z\rangle + |\lambda|^2\|z\|^2 \geqslant 0.$$

Write $\lambda = |\lambda|e^{i\theta}$, fix θ, divide by $|\lambda|$ and let $|\lambda|$ go to zero to conclude

$$-2\Re\langle x - \tilde{z}, e^{i\theta}z\rangle \geqslant 0.$$

Since θ is arbitrary we must have $\langle x - \tilde{z}, z\rangle = 0$. □

Definition 5.1.15. Let $S \subseteq \mathcal{H}$ be a subset. Then $S^\perp := \{x \in \mathcal{H} : \langle x, y\rangle = 0, \forall y \in S\}$.

Proposition 5.1.16. Let $S \subseteq \mathcal{H}$ be a subset. Then the following holds.

1. $S^\perp$ is a closed subspace.
2. $S^{\perp\perp} := (S^\perp)^\perp$ is the closure of linear span of S.
3. $S \cap S^\perp \subseteq \{0\}$. If $0 \in S$ then $S \cap S^\perp = \{0\}$.

Proof. Obvious. □

Theorem 5.1.17 (Projection theorem)**.** *Let $\mathcal{H}_0 \subseteq \mathcal{H}$ be a closed subspace. Then every $x \in \mathcal{H}$ can be written uniquely as $y+z$ where $y \in \mathcal{H}_0, z \in \mathcal{H}_0^\perp$. The mapping $P_{\mathcal{H}_0} : x \mapsto y$ is a bounded linear map from $\mathcal{H}$ to itself so that $P_{\mathcal{H}_0}^2 = P_{\mathcal{H}_0}$.*

Proof. Let $y = \arg\min\{\|x-u\| : u \in \mathcal{H}_0\}$ and $z = x-y \in \mathcal{H}_0^{\perp}$ by proposition (5.1.14). To see uniqueness of the decomposition note that if $x = y_1 + z_1 = y_2 + z_2$ with $y_1, y_2 \in \mathcal{H}_0, z_1, z_2 \in \mathcal{H}_0^{\perp}$, then $y_1 - y_2 = z_2 - z_1 \in \mathcal{H}_0 \cap \mathcal{H}_0^{\perp} = \{0\}$. Clearly $P_{\mathcal{H}_0} : x \mapsto y$ is linear. To see it is bounded let us calculate $\|x\|^2$, keeping in mind $\langle y, z\rangle = 0$.

$$\|x\|^2 = \langle y+z, y+z\rangle = \langle y, y\rangle + \langle z, z\rangle = \|y\|^2 + \|z\|^2 \geqslant \|y\|^2 = \|P_{\mathcal{H}_0}(x)\|^2.$$

Therefore $P_{\mathcal{H}_0}$ is bounded with norm bounded by 1. If $\mathcal{H}_0 \neq \{0\}$ then $\|P_{\mathcal{H}_0}\| = 1$. □

Theorem 5.1.18 (Riesz Representation Theorem)**.** *Let $\phi \in \mathcal{H}^*$, then there is unique $u_\phi \in \mathcal{H}$ so that $\phi(v) = \langle u_\phi, v\rangle$. Moreover $\|\phi\| = \|u_\phi\|$. The mapping $\phi \mapsto u_\phi$ gives a conjugate linear isometry from $\mathcal{H}^*$ to $\mathcal{H}$.*

Proof. Let $\mathcal{H}_0 = \ker\phi$. Note that $\phi = 0$ if and only if $\ker\phi = \mathcal{H}$. So, if $\phi = 0$ we can take $u_\phi = 0$. Let us now consider the case $\phi \neq 0$. Then $\mathcal{H}_0$ is a proper subspace. So there exists $v \in \mathcal{H}_0^{\perp}$ satisfying $\phi(v) = 1$. By the first isomorphism theorem of linear algebra $\mathcal{H}_0^{\perp} = \mathbb{C}v$. Let $u_\phi = \frac{v}{\|v\|^2}$, then

$$\langle u_\phi, w\rangle = \begin{cases} 0, \text{if} & w \in \mathcal{H}_0 \\ 1, \text{if} & w \in \mathcal{H}_0^{\perp}. \end{cases}$$

Thus $\phi(w) = \langle u_\phi, w\rangle, \forall w$. An application of Cauchy-Schwarz inequality yields $\|\phi\| = \|u_\phi\|$. □

Definition 5.1.19. Let $\mathcal{H}$ be a Hilbert space.

1. Orthogonal set: A subset $S \subseteq \mathcal{H}$ is said to be orthogonal if every element of S is nonzero and $v, w \in S, v \neq w$ implies $\langle v, w\rangle = 0$.

2. Orthonormal set: A subset $S \subseteq \mathcal{H}$ is said to be orthonormal if it is orthogonal and every element of S has norm one.

3. Orthonormal basis: A maximal with respect to inclusion orthonormal set is called an orthonormal basis to be abbreviated as O.N.B. It exists by a simple application of Zorn's lemma.

4. An orthonormal set S is said to be complete if $\mathcal{H} = \overline{\operatorname{Span} S}$.

Definition 5.1.20. Let X be a set and $f : X \to \mathbb{R}_{\geqslant 0}$ be a function. Let $\mathcal{F} := \{F \subseteq X : F \text{ is a finite set}\}$. This is directed by inclusion. The limit of the net $\{s_F := \sum_{x\in F} f(x)\}_{F\in\mathcal{F}}$ if exists is denoted by $\sum_{x\in X} f(x)$.

Theorem 5.1.21 (Bessel's inequality)**.** *Let $\mathfrak{B}$ be an orthonormal set. Then for all $v \in \mathcal{H}$ we have $\sum_{u\in\mathfrak{B}} |\langle u, v\rangle|^2 \leqslant \|v\|^2$.*

Proof. Let $F \subseteq \mathfrak{B}$ be a finite subset. Then $\{\langle u, v\rangle u : u \in F\} \cup \{v - \sum_{u\in F}\langle u, v\rangle u\}$ is an orthogonal set and by exercise (5.1.22) we have

$$\sum_{u\in F} \|\langle u, v\rangle u\|^2 + \|v - \sum_{u\in F}\langle u, v\rangle u\|^2 = \|v\|^2.$$

Therefore $\sum_{u\in F} \|\langle u, v\rangle u\|^2 \leqslant \|v\|^2$. The net $F \mapsto \sum_{u\in F} \|\langle u, v\rangle u\|^2$ is a montone net bounded by $\|v\|^2$. Hence it converges to $\sum_{u\in\mathfrak{B}} |\langle u, v\rangle|^2 \leqslant \|v\|^2$. □

Exercise 5.1.22. Let S be a finite orthogonal set. Then $\| \sum_{u\in S} u\|^2 = \sum_{u\in S} \|u\|^2$.

Proposition 5.1.23. Every orthonormal set can be extended to a orthonormal basis.

Proof. Let $\mathfrak{B}$ be an orthonormal set. Consider the partially ordered set $\mathcal{P} := \{\mathfrak{B}' : \mathfrak{B}' \supset \mathfrak{B}, \mathfrak{B}'$ is an O.N.B $\}$ ordered by inclusion. Clearly every chain in this partially ordered set has an upper bound it has a maximal element $\mathfrak{B}'$. This gives an orthonormal basis containing $\mathfrak{B}$. □

Lemma 5.1.24. *Let S be an orthonormal set and $x \in \mathcal{H}$, then the orthogonal projection of x on span of S is given by $\sum_{v\in S}\langle v, x\rangle v$.*

Proof. Note that $\langle x - \sum_{v\in S}\langle v, x\rangle v, w\rangle = 0, \forall w \in S$. Therefore

$$\begin{aligned} \|x - \sum_{v\in S} \lambda_v v\|^2 &= \|x - \sum_{v\in S}\langle v, x\rangle v + \sum_{v\in S}(\lambda_v + \langle v, x\rangle)v\|^2 \\ &= \|x - \sum_{v\in S}\langle v, x\rangle v\|^2 + \sum_{v\in S} |(\lambda_v + \langle v, x\rangle)|^2 \quad \text{[By pythagoras} \\ &\geqslant \|x - \sum_{v\in S}\langle v, x\rangle v\|^2 \end{aligned} \tag{5.1}$$

Thus $\sum_{v\in S}\langle v, x\rangle v = \arg\min\{\|x - u\| : u \in \operatorname{Span} S\}$. □

Proposition 5.1.25. Let $S \subseteq \mathcal{H}$ be an orthonormal set then the following are equivalent.

1. S is an orthonormal basis.
2. S is complete.
3. Parseval's relation: For all $x \in \mathcal{H}$, $\|x\|^2 = \sum_{v\in S} |\langle v, x\rangle|^2$

Proof. (1) $\implies$ (2) : Let $\mathcal{H}_0$ be the closed linear span of S. If $\mathcal{H}_0 \subsetneq \mathcal{H}$, then choose $v \in \mathcal{H} \setminus \mathcal{H}_0$. The vector $w := v - P_{\mathcal{H}_0} v$ must be non-zero because otherwise $v = P_{\mathcal{H}_0} v \in \mathcal{H}_0$. Since $w \in \mathcal{H}_0^{\perp}$, $S \cup \{\frac{w}{\|w\|}\}$ is an orthonormal basis properly containing S. This contradicts maximality of S!

$(2) \implies (3)$: Let $x \in \mathcal{H}$. Then for any finite set $F \subseteq S$, $(x - \sum_{v \in F} \langle v, x \rangle v) \perp v, \forall v \in F$. Therefore by pythagoras' theorem

$$\|x\|^2 = \sum_{v \in F} |\langle v, x \rangle|^2 + \|x - \sum_{v \in F} \langle v, x \rangle v\|^2 \tag{5.2}$$

Using completeness of S, for each $\epsilon > 0$ we get $v_1, \ldots, v_{n(\epsilon)} \in S$ and scalars $\lambda_1, \ldots, \lambda_{n(\epsilon)}$ so that $\|x - \sum_{j=1}^{n(\epsilon)} \lambda_j v_j\| < \epsilon$. If we call the finite set $\{v_1, \ldots, v_{n(\epsilon)}\}$, F_ϵ then by (5.1)

$$\|x - \sum_{v \in F_\epsilon} \langle v, x \rangle v\|^2 \leqslant \|x - \sum_{j=1}^{n(\epsilon)} \lambda_j v_j\|^2 < \epsilon^2 \tag{5.3}$$

Therefore the net $F \mapsto x - \sum_{v \in F_\epsilon} \langle v, x \rangle v$ defined on the directed set of finite subsets of S converges to 0. In other words the second term in (5.2) converges to 0. This proves $\|x\|^2 = \lim_F \sum_{v \in F} |\langle v, x \rangle|^2$.

$(3) \implies (1)$: If possible let $x \in \mathcal{H} \setminus S$ be such that $\{x\} \cup S$ be orthonormal. Then $\langle v, x \rangle = 0, \forall v \in S$. Therefore $\|x\|^2 = \sum_{v \in S} |\langle v, x \rangle|^2 = 0$, a contradiction to orthonormality of $\{x\} \cup S$. □

Corollary 5.1.26 (Abstract Fourier Expansion)**.** Let S be an orthonormal basis. Then for all $x \in \mathcal{H}$ we have $x = \sum_{v \in S} \langle v, x \rangle v$.

Proof. Since $\|x\|^2 = \lim_F \sum_{v \in F} |\langle v, x \rangle|^2$, from (5.2) we have $\lim_F \|x - \sum_{v \in F} \langle v, x \rangle v\| = 0$ or equivalently $x = \lim_F \sum_{v \in F} \langle v, x \rangle v =: \sum_{v \in S} \langle v, x \rangle v$. □

Week 6

Problems

6.1 Monday, 13th February 2023

Exercise 6.1.1. Let E be a Banach space and F a finite dimensional subspace. Show that F is closed.

Exercise 6.1.2. Let E be a finite dimensional Banach space. Can you give a dense proper subspace of E?

Exercise 6.1.3. Let E be an infinite dimensional Banach space. Give a dense proper subspace of E.

Exercise 6.1.4. Let E be a Banach space and F a closed subspace. We say F is algebraically complemented if there is another closed subspace F' such that $F \oplus F' = E$. Suppose F is finite dimensional. Then show that F is algebraically complemented.

Exercise 6.1.5. Let E be a Banach space and F a closed subspace. We say F is topologically complemented if it is algebraically complemented and the norm on E is equivalent to the norm on the ℓ_1-sum of F and F' where F, F' are endowed with norms obtained from E as its subspaces. Show that if a closed subspace is algebraically complemented then it is topologically complemented.

Exercise 6.1.6. Let E be a Banach space and $\phi : E \to \mathbb{K}$ be an unbounded linear functional then show that $\ker \phi$ is dense in E.

Exercise 6.1.7. Let E be a Banach space and $\phi : E \to \mathbb{K}$ be a linear map. If $\ker \phi$ is a dense proper subspace then show that ϕ must be unbounded.

Exercise 6.1.8. Let E be a Banach space and $\phi : E \to \mathbb{K}$. Then $\ker \phi$ is closed iff ϕ is continuous.

Solution. We only need to show the only if part. Let $q : E \to E/\ker \phi$ be the quotient map. Let $\tilde{\phi} : E/\ker \phi \to \mathbb{K}$ be the induced map. It is continuous because $E/\ker \phi$ is one dimensional and any linear map on a finite dimensional space is continuous. Then being a composition of continuous maps $\phi = \tilde{\phi} \circ q$ is continuous. □

Exercise 6.1.9. Show that there is a bounded linear map $L : \ell_\infty \to \mathbb{R}$ such that

1. $\liminf \underline{\mathbf{x}} \leqslant L(\underline{\mathbf{x}}) \leqslant \limsup \underline{\mathbf{x}}$.
2. $L(\underline{\mathbf{x}}) = \lim x_n$ if $L(\underline{\mathbf{x}}) = \{x_n\}$ is a convergent sequence.
3. $L(\underline{\mathbf{x}}) = L(S(\underline{\mathbf{x}}))$ where $S : \ell_\infty \to \ell_\infty$ is the shift operator given by $S(\underline{\mathbf{x}})_n = (\underline{\mathbf{x}})_{n+1}$.

Exercise 6.1.10. Let E, F be Banach spaces and $T_n \in \mathcal{L}(E; F)$ be such that for all $x \in E$, the sequence $\{T_n(x)\}$ is convergent. Then show that $\sup_n \|T_n\| < \infty$. Let $T(x) := \lim T_n(x)$. Then show that $T \in \mathcal{L}(E; F)$. If $x_n \to x$, then show that $T_n(x_n) \to T(x)$.

6.2 Wednesday, 15th February 2023

Exercise 6.2.1. Show that for each n, k there exists $C_{n,k} > 0$ such that for all polynomials P of degree less than or equal to n, in k variables with $\mathbb{K}$ coefficients we have

$$\sup_{x \in B(0;r) \subseteq \mathbb{R}^k} |P(x)| \leqslant C_{n,k} \int_{B(0;r)} \frac{|P(x)|}{\mathrm{Vol}(B(0;r))} dx.$$

Exercise 6.2.2. Given any two isomorphic Banach spaces E, F define their Banach Mazur distance as

$$\delta_{BM}(E, F) := \{\|T\|.\|T^{-1}\| : T \in \mathcal{L}(E, F) \text{ is invertible with } T^{-1} \in \mathcal{L}(F, E)\}$$

Then show that $\delta_{BM}(E, F) \geqslant 1$ and $\delta_{BM}(E, F) = 1$ along with $\dim E < \infty$ implies E, F are linearly isometrically isomorphic.

Solution. First note that since $\|\lambda.T\|\|(\lambda.T)^{-1}\| = \|T\|\|T^{-1}\|$ we have

$$\delta_{BM}(E, F) = \{\|T^{-1}\| : \|T\| = 1, T \in \mathcal{L}(E, F) \text{ is an isomorphism}\}.$$

Let $\delta_{BM}(E, F) = 1$. Then there exists a sequence $T_n \in \mathcal{L}(E, F)$ of norm 1 such that $\|T_n^{-1}\| \to 1$. Since E is finite dimensional the unit ball of $\mathcal{L}(E, F)$ is compact. Therefore along a subsequence T_n converges to some T. By passing to this subsequence we can assume $T_n \to T$.

Claim T must be one to one:

Proof of claim. Suppose $Tx = 0$. Then using $\lim T_n = T$ we get $Tx = \lim T_n x = 0$. Using $x = \lim T_n^{-1} T_n(x)$ we get

$$\|x\| = \lim \|T_n^{-1} T_n x\| \leqslant \limsup \|T_n^{-1}\| \|T_n x\| \leqslant M \lim \|T_n x\| = 0$$

where $M = \limsup \|T_n^{-1}\| < \infty$ because $\|T_n^{-1}\| \to 1$. □

Therefore T is an isomorphism. Since $S \mapsto S^{-1}$ is a continuous map from $\mathcal{L}(E,F)$ to $\mathcal{L}(F,E)$ and norm is a continuous map we must have $\|T_n^{-1}\| \to \|T^{-1}\|$. But $\lim \|T_n - 1\| = 1$. Therefore $\|T^{-1}\| = 1$ as well. So we have both $\|T\| = 1 = \|T^{-1}\|$. In other words T is the required isometry between E and F. □

Exercise 6.2.3. Let E be a separable, then E is a quotient of ℓ_1.

6.3 Friday, 17th February

Exercise 6.3.1. Let $F \subseteq E$ be a closed subspace of a Banach space. Show that $\Phi : (E/F)^* \to F^\perp := \{x^* \in E^* : \langle x^*, x\rangle = 0, \forall x \in F\}$ given by $\Phi(\phi)(x) = \phi([x])$ is a linear isometric one to one onto map.

Exercise 6.3.2. Let $F \subseteq E$ be a closed subspace of a Banach space. Define $\Psi : F^* \to E^*/F^\perp$ as follows: given $\phi \in F^*$ by Hahn Banach obtain a norm preserving extension $\tilde{\phi}$. Define $\Psi(\phi) = [\tilde{\phi}]$. Show that Ψ is a linear isometric isomorphism.

Exercise 6.3.3. Let E be a reflexive Banach space. Show that for all $x^* \in E^*, \exists x \in E, \|x\| = 1, x^*(x) = \|x^*\|$.

Exercise 6.3.4. Goal of this exercise is showing the collection of continuous nowhere differentiable functions is a dense G_δ subset of $C[0,1]$. This exercise is from Pedersen's Analysis Now.

1. Let $\mathcal{F}_n = \{f \in C[0,1] : \exists x_f \in [0,1]$, such that $\forall y \in [0,1], |f(y) - f(x_f)| \leqslant n|y - x_f|, \}$. Then show that $\mathcal{F}_n$ is closed.
2. Let $f \in C[0,1]$ be differentiable at x. Then show that $f \in \cup_n \mathcal{F}_n$.
3. Finally show that $\mathcal{F}_n$ has empty interior.
4. Conclude that no where differentiable continuous functions form a dense G_δ subset of $C[0,1]$.

6.4 Monday, 20th February

Exercise 6.4.1. Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ be a sesquilinear form. If there exists a positive constant C such that

$$|T(u,v)| \leqslant C\|u\|\|v\|, \forall u, v \in \mathcal{H}.$$

Then there is a unique bounded linear map $\widetilde{T} \in B(\mathcal{H})$ such that $\|\widetilde{T}\| \leqslant C$ and

$$T(u,v) = \langle \widetilde{T}(u), v\rangle, \forall u, v \in \mathcal{H}.$$

Exercise 6.4.2. If we have Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, and a sesquilinear map $B : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{K}$ such that

$$|B(u,v)| \leqslant C\|u\|\|v\|, \forall u \in \mathcal{H}_1, \forall v \in \mathcal{H}_2$$

where C is a positive constant then there exists a bounded linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ of norm less than or equal to C and

$$B(u,v) = \langle T(u), v\rangle, \forall u \in \mathcal{H}_1, \forall v \in \mathcal{H}_2.$$

Exercise 6.4.3. Let $x, y : [0,1] \to \mathbb{R}$ be C^1-functions such that $\|\frac{dx}{dt}\|_2^2 + \|\frac{dy}{dt}\|_2^2 = \ell^2$, then $|\int_0^1 y(t)\frac{dx}{dt}dt| \leqslant \frac{\ell^2}{4\pi}$.

Solution to Exercise 6.4.3. Let $x'(t) = \frac{dx}{dt}$ and $y'(t) = \frac{dy}{dt}$. Then

$$\begin{aligned}\hat{x'}(n) &= \int_0^1 e^{-2\pi i n t}\frac{dx}{dt}dt \\ &= e^{-2\pi i n t}x(t)|_{t=0}^{t=1} + 2\pi i n\int_0^1 e^{-2\pi i n t}x(t)dt \\ &= 2\pi i n\hat{x}(n).\end{aligned}$$

Similarly $\hat{y'}(n) = 2\pi i n\hat{y}(n)$. Therefore

$$\ell^2 = \|\frac{dx}{dt}\|^2 + \|\frac{dy}{dt}\|^2 = 4\pi^2\sum n^2(|\hat{x}(n)|^2 + |\hat{y}(n)|^2).$$

$$\begin{aligned}\left|\int_0^1 y\frac{dx}{dt}dt\right| &= |\langle y, x'\rangle| \\ &= |\sum_n 2\pi n\hat{y}(n)\hat{x}(n)| \\ &\leqslant 2\pi\sqrt{\sum_{n\neq 0}|\hat{y}(n)|^2}\sqrt{\sum_{n\neq 0}n^2|\hat{x}(n)|^2}, \quad \text{by Cauchy-Schwarz inequality} \\ &\leqslant \pi(\sum_{n\neq 0}|\hat{y}(n)|^2 + \sum_{n\neq 0}n^2|\hat{x}(n)|^2) \\ &\leqslant \frac{1}{4\pi}(4\pi^2\sum n^2(|\hat{x}(n)|^2 + |\hat{y}(n)|^2)) \\ &= \frac{\ell^2}{4\pi}\end{aligned}$$

□

Exercise 6.4.4 (Lax-Milgram)**.** The bilinear form T is called coercive if $\exists a > 0$ such that $T(u,u) \geqslant a\|u\|^2, \forall u \in \mathcal{H}$. By exercise (6.4.1) we know that there exists $\widetilde{T} \in B(\mathcal{H})$ such that $T(u,v) = \langle\widetilde{T}(u), v\rangle$. If T is given to be coercive.

(i) Show that $\widetilde{T}$ is one to one.

(ii) Let $\mathfrak{Ran}$ be the range of $\widetilde{T}$. Consider $S : \mathfrak{Ran} \to \mathcal{H}$ given by $S(u) = v$ where $u = \widetilde{T}(v)$. Show that S is bounded. and using this show that $\mathfrak{Ran}$ is closed.

(iii) Show that $\widetilde{T}$ is onto i.e., $\mathfrak{Ran} = \mathcal{H}$.

(iv) Conclude given $\phi \in \mathcal{H}$ there exists unique $u \in \mathcal{H}$ such that $T(u, v) = \langle \phi, v \rangle, \forall v \in \mathcal{H}$.

Solution to Exercise 6.4.4. (i) The map $\widetilde{T}$ is one to one because if $\widetilde{T}(u) = 0$, then

$$0 \leqslant a\|u\|^2 \leqslant |T(u, u)| = |\langle \widetilde{T}(u), u \rangle| = 0.$$

Thus $u = 0$.

(ii) The map S is well defined because $\widetilde{T}$ is one to one.

$$0 \leqslant a\|v\|^2 \leqslant |T(v, v)| = |\langle \widetilde{T}(v), v \rangle| = |\langle u, v \rangle| \leqslant \|u\|\|v\|. \tag{6.1}$$

Therefore $\|v\| = \|S(u)\| \leqslant 1/a\|u\|$. So, S is a bounded linear map. Let $\{\widetilde{T}(u_n)\}$ be a Cauchy sequence converging to w, then $\{S(\widetilde{T}(u_n))\}$ is also Cauchy. That is $\{u_n\}$ is Cauchy. Let u be the limit of $\{u_n\}$. Then $w = T(u) \in \mathfrak{Ran}$. This shows that $\mathfrak{Ran}$ is closed.

(iii) Let u be orthogonal to $\mathfrak{Ran}$, then

$$0 \leqslant a\|u\|^2 \leqslant |T(u, u)| = |\langle \widetilde{T}(u), u \rangle| = 0.$$

Thus u must be zero. This shows $\widetilde{T}$ must be onto.

(iv) Let $u = S(\phi)$, then $\widetilde{T}(u) = \phi$ and

$$T(u, v) = \langle \widetilde{T}(u), v \rangle = \langle \phi, v \rangle.$$

□

Exercise 6.4.5. Let $(\Omega, \mathfrak{S}, \mu)$ be a probability space and $\mathfrak{S}' \subseteq \mathfrak{S}$ a sub-σ-algebra. Let f be a nonnegative measurable L_1 function. Let $L_2(\mathfrak{S}')$ be the space of square integrable $\mathfrak{S}'$ measurable functions. Then $L_2(\mathfrak{S}') \subseteq L_2(\mathfrak{S})$ is a closed subspace. Let P be the corresponding projection. Show that

1. If $0 \leqslant f \leqslant C$ then $\exists N \in \mathfrak{S}', \mu(N) = 0$ and a $\mathfrak{S}'$ measurable g such that on $N^c, 0 \leqslant g \leqslant C$ and $g = Pf$ a.e. Such a g will be called a version of Pf.

2. $$\int_A f d\mu = \int_A P f d\mu, \forall A \in \mathfrak{S}'.$$

3. Let $f_n = f \wedge n$, then $\exists N \in \mathfrak{S}', \mu(N) = 0$ such that outside N, each Pf_n has a version g_n such that $0 \leqslant g_n \leqslant n$ and $g_n \leqslant g_{n+1}, \forall n \geqslant 1$. Let $g = \lim g_n$. Show that
$$\int_A f d\mu = \int_A g d\mu, \forall A \in \mathfrak{S}'. \tag{6.2}$$
Such a g is called the conditional expectation of f given $\mathfrak{S}'$ and is denoted by $\mathbb{E}(f|\mathfrak{S}')$. This is an $\mathfrak{S}'$ measurable integrable function unique upto a μ null set.

Solution to Exercise 6.4.5. (2) We know that $Pf \in L_2(\Omega, \mathfrak{S}', \mu)$. Therefore if $A \in \mathfrak{S}'$ then
$$\int_A Pf d\mu = \langle 1_A, Pf \rangle = \langle 1_A, f \rangle = \int_A f d\mu.$$
(1) Let $A_n = \{\omega : Pf(\omega) \leqslant -1/n\} \in \mathfrak{S}'$, then
$$0 \leqslant \int_{A_n} f = \int_{A_n} Pf d\mu \leqslant (-1/n)\mu(A_n) \leqslant 0.$$
Therefore $\mu(A_n) = 0$ and consequently $\mu(\omega : Pf(\omega) < 0) = \lim_{n\to\infty} \mu(A_n) = 0$. Similarly considering $P(C - f)$ we conclude that $\mu(Pf \leqslant C) = 1$.

(3) $f_n \leqslant f_{n+1}$ implies there exists N_n such that $\mu(N_n) = 0, N_n \in \mathfrak{S}'$ and outside $N_n, Pf_n \leqslant Pf_{n+1}, \forall n$. Let $N = \cap N_n$, then $N \in \mathfrak{S}', \mu(N) = 0$ and outside N, $Pf_n \nearrow \mathbb{E}(f|\mathfrak{S}')$. To see (6.2) note that
$$\int_A f d\mu = \lim_n \int_A f_n d\mu = \lim_n \int_A Pf_n d\mu = \int_A \mathbb{E}(f|\mathfrak{S}'), \forall A \in \mathfrak{S}'.$$
The first and the last equality follows from the monotone convergence theorem. □

Week 7

Compact Operators on Banach Spaces

7.1 Riesz Lemma

Lemma 7.1.1 (Riesz lemma)**.**

Proposition 7.1.2. A normed linear space is finite dimensional provided it is locally compact.

7.2 Compact Operators

Definition 7.2.1 (Compact operator)**.**

Definition 7.2.2. A linear operator is called finite rank if its range is a finite dimensional subspace.

Theorem 7.2.3. *Let $T \in B(\mathcal{H})$, then T is a compact operator if and only if T is a norm limit of finite rank operators.*

Proof. **Only if part:** Let T be a compact operator. Therefore given $\epsilon > 0$, there exists $y_1, \cdots, y_{n_\epsilon}$ such that $\overline{T(B(0,1))} \subseteq \cup_{j=1}^{n_\epsilon} B(y_j, \epsilon)$. Let $\{e_\alpha\}_{\alpha \in A}$ be an o.n.b for $\mathcal{H}$. Then there exists a finite subset F of A such that

$$\sum_{\alpha \notin F} |\langle y_j, e_\alpha \rangle|^2 \; < \; \epsilon^2, \forall j = 1, \cdots, n_\epsilon. \tag{7.1}$$

Let y be an element of the norm closure of $T(B(0,1))$. Then there exists y_j such that

$$\sum_{\alpha \notin F} |\langle y - y_j, e_\alpha \rangle|^2 \leqslant \sum_{\alpha \in A} |\langle y - y_j, e_\alpha \rangle|^2 = \|y - y_j\|^2 < \epsilon^2 \tag{7.2}$$

Therefore,

$$\sum_{\alpha \notin F} |\langle y, e_\alpha \rangle|^2 \ < \ 4\epsilon^2. \tag{7.3}$$

Let P_F be the orthogonal projection on the span of $\{e_\alpha : \alpha \in F\}$ and $T_F = P_F T$. By proposition (8.1.1) T_F is a compact operator. Let $x \in B(0,1)$ and $y = T(x) \in \overline{T(B(0,1))}$. By (7.3) we see that $\|T(x) - T_F(x)\| < 2\epsilon$. Therefore $\|T - T_F\| < 2\epsilon$.

If part: Let $\{T_n\}$ be a sequence of finite rank operators such that $\|T_n - T\| \to 0$. Let $\{u_\alpha\}$ be a weakly convergent net with u as its weak limit, i.e., $\langle v, u_\alpha \rangle \to \langle v, u \rangle, \forall v \in \mathcal{H}$. The set $\{u_\alpha\}$ is weakly bounded and hence by corollary (4.2.2) is norm bounded say by $M > 1$. Find N such that $\|T_n - T\| < \frac{\epsilon}{3M}$ whenever $n \geqslant N$. Let γ be such that $\|T_N u_\alpha - T_N u_\beta\| < \frac{\epsilon}{3M}$ provided $\alpha, \beta \succ \gamma$. Then for such α, β,

$$\|Tu_\alpha - Tu_\beta\| \leqslant \|Tu_\alpha - T_N u_\alpha\| + \|Tu_\beta - T_N u_\beta\| + \|T_N u_\alpha - T_N u_\beta\| \leqslant \epsilon.$$

Thus $\{T(u_\alpha)\}$ is a Cauchy net hence convergent. $\square$

Week 8

Spectral Theorem for Compact Operators on Hilbert Spaces

In this chapter we will begin with the spectral theorem for compact self adjoint operators. Learn singular value decomposition. We will also discuss the basic Fredholm theory.

8.1 Spectral Theorem for Compact Operators

We have already encountered the definition of a compact operator and the spectral theory of compact operators on Banach spaces. Now we focus on compact operators on Hilbert spaces.

Proposition 8.1.1. Let $T \in B(\mathcal{H})$ then T is compact if and only if T converts weakly convergent nets to norm convergent nets. That is

$$(\langle v, u_\alpha \rangle \to \langle v, u \rangle, \forall v \in \mathcal{H}) \implies \|T(u_\alpha) - T(u)\| \to 0.$$

Proof. Let $\{u_\alpha\}_{\alpha \in A}$ be a weakly convergent net with u as its limit. The net $\{T(u_\alpha)\}$ weakly converges to $T(u)$ because

$$\langle v, T(u_\alpha) \rangle = \langle T^*(v), u_\alpha \rangle \to \langle T^*(v), u \rangle = \langle v, T(u) \rangle$$

In order to utilize the hypothesis that T is a compact operator note that the set $\{u_\alpha : \alpha \in A\}$ is weakly bounded. Hence by corollary (4.2.2) it is norm bounded. So there exists M such that $\sup\{\|u_\alpha\| : \alpha \in A\} < M$. Since T is compact any subnet of $\{T(u_\alpha)\}$ has a convergent subnet and the limit must be $T(u)$, because $\{T(u_\alpha)\}$ weakly converges to $T(u)$. Since the limit of the convergent subnet of any given subnet does not depend on the net the original net must be convergent with the same limit, i.e., $\|T(u_\alpha) - T(u)\| \to 0$.

Conversely, let $\{T(u_\alpha)\}$ be a net in $T(B(0,1))$. By Banach-Aloglu theorem we can conclude that $\{u_\alpha\}$ has a convergent subnet. Then the corresponding subnet $\{T(u_\alpha)\}$ converges. This shows that $T(B(0,1))$ is relatively compact or equivalently has compact closure. □

Theorem 8.1.2. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space and $T \in B(\mathcal{H})$ be a nonzero self-adjoint compact operator, then*

$$\Lambda_+ = \sup\{\langle u, Tu\rangle : \|u\| = 1\} = \sup\{\langle u, Tu\rangle : \|u\| \leqslant 1\}$$

$$\Lambda_- = \inf\{\langle u, Tu\rangle : \|u\| = 1\} = \inf\{\langle u, Tu\rangle : \|u\| \leqslant 1\}$$

are attained. Let u_+, u_- be the vectors where Λ_+, Λ_- are attained, then at least one of the following holds,

$$Tu_\pm = \Lambda_\pm u_\pm.$$

Proof. Let $F(u) = \langle u, Tu\rangle$, then this is a real valued function because,

$$\overline{F(u)} = \langle Tu, u\rangle = \langle u, T^*u\rangle = \langle u, Tu\rangle = Fu.$$

Also for $\|u\| \leqslant 1$, $|F(u)| \leqslant \|u\|^2\|T\| \leqslant \|T\|$. Therefore $\Lambda_\pm$ makes sense. Let $\{u_n\}$ be a sequence such that $\|u_n\| \leqslant 1$ and $F(u_n) \to \Lambda_+$. Since a Hilbert space is reflexive by Banach-Alaoglu theorem its unit ball is weakly compact the sequence $\{u_n\}$ has a weakly convergent subsequence. Without loss of generality we can assume that $u_n \to u_+$, weakly. Then,

$$\begin{aligned} |F(u_n) - F(u_+)| &= |\langle u_n, Tu_n\rangle - \langle u_+, Tu_+\rangle| \\ &\leqslant |\langle u_n, Tu_n - Tu_+\rangle| + |\langle u_n - u_+, Tu_+\rangle| \\ &\leqslant \|Tu_n - Tu_+\| + |\langle u_n - u_+, Tu_+\rangle| \\ &\to 0. \end{aligned}$$

Since T is sompact the first term goes to zero and the second term goes to zero because $\{u_n\}$ weakly converges to u_+. Therefore $F(u_+) = \lim F(u_n) = \Lambda_+$. Let $\{e_n : n \geqslant 1\}$ be an infinite orthonormal set. Then $\{e_n\}$ weakly converges to zero, hence $\{T(e_n)\}$ converges to zero in norm. Therefore $\{F(e_n)\}$ converges to zero. Thus $\Lambda_+ \geqslant 0$. If $\|u_+\| < 1$ there exists $\epsilon > 0$ such that $\|(1+\epsilon)u_+\| = 1$, and $F((1+\epsilon)u_+) = (1+\epsilon)F(u_+) \geqslant F(u_+)$. Similarly we obtain u_- such that $F(u_-) = \Lambda_-$.

$\Lambda_\pm$ both can not be zero: Suppose that $\Lambda_+ = \Lambda_- = 0$. Then for any u of unit norm, $F(u) = 0$. Thus for any u, we get $\langle u, Tu\rangle = 0$. Then by polarization we get

$$2\langle v, Tu\rangle = \langle u+v, T(u+v)\rangle + i\langle u+iv, T(u+iv)\rangle = 0.$$

Therefore $T = 0$ a contradiction to $T \neq 0$! □

Without loss of generality we assume that $\Lambda_+ \neq 0$. Then $\langle u_+, Tu_+\rangle = \Lambda_+ > 0$. Therefore, $T(u_+) \neq 0$.

Claim: $v \in \mathcal{H}, \|v\| = 1, v \perp u_+ \implies v \perp Tu_+$

Proof of Claim: Let $v_\theta = (\mathrm{Cos}\theta)v + (\mathrm{Sin}\theta)u_+$, then $\|v_\theta\| \leqslant 1$ and

$$\begin{aligned} F(v_\theta) &= \mathrm{Cos}^2\theta.F(v) + \mathrm{Sin}^2\theta.F(u_+) + \mathrm{Cos}\theta\mathrm{Sin}\theta\langle v, Tu_+\rangle \\ &\quad +\mathrm{Sin}\theta\mathrm{Cos}\theta\langle u_+, Tv\rangle \\ &= \mathrm{Cos}^2\theta F(v) + \mathrm{Sin}^2\theta F(u_+) + \mathrm{Sin}2\theta\Re\langle v, Tu_+\rangle \end{aligned}$$

We know that the function $\theta \mapsto F(v_\theta)$ attains its maximum at $\theta = \pi/2$. Therefore

$$\frac{dF(v_\theta)}{d\theta}|_{\theta=\pi/2} = \Re\langle v, Tu_+\rangle = 0.$$

Instead of v if we put $\sqrt{-1}v$ we obtain $\Im\langle v, Tu_+\rangle = 0$. Therefore $\langle v, Tu_+\rangle = 0$. □

Thus, $Tu_+ \in u_+^{\perp\perp} = \mathbb{C}u_+$. Let $Tu_+ = \lambda u_+$, and

$$\Lambda_+ = F(u_+) = \langle u_+, Tu_+\rangle = \lambda\|u_+\|^2 = \lambda.$$

If $\Lambda_- \neq 0$ we similarly conclude that $Tu_- = \Lambda_- u_-$. □

Lemma 8.1.3. *Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then*

$$\|T\| = \sup\{\frac{|\langle u, Tu\rangle|}{\|u\|^2} : \|u\| \neq 0\}. \tag{8.1}$$

Proof. Let M be the right hand side of 8.1. By Cauchy-Schwarz inequality we see that $M \leqslant \|T\|$. Let $u, v \in \mathcal{H}$, then

$$\begin{aligned} \langle u+v, T(u+v)\rangle &= \langle u, Tu\rangle + \langle u, Tv\rangle + \langle v, Tu\rangle + \langle v, Tv\rangle \\ \langle u-v, T(u-v)\rangle &= \langle u, Tu\rangle - \langle u, Tv\rangle - \langle v, Tu\rangle + \langle v, Tv\rangle \end{aligned}$$

Subtracting and taking absolute values we get

$$2|\langle u, Tv\rangle + \langle v, Tu\rangle| = |\langle u+v, T(u+v)\rangle - \langle u-v, T(u-v)\rangle| \tag{8.2}$$

If T is the zero operator then clearly $\|T\| \leqslant M$. So, we can assume $T \neq 0$. Let u be an arbitrary unit vector such that $Tu \neq 0$. Let $v = \frac{Tu}{\|Tu\|}$. Then, $\langle u, Tv\rangle = \langle Tu, v\rangle = \|Tu\|$. Putting these in 8.2 we get

$$\begin{aligned} 4\|Tu\| &= |\langle u+v, T(u+v)\rangle - \langle u-v, T(u-v)\rangle| \\ &\leqslant M(\|u+v\|^2 + \|u-v\|^2) \\ &= M2(\|u\|^2 + \|v\|^2) && \text{[by parallelogram identity} \\ &= 4M && \text{[since } \|u\| = \|v\| = 1. \end{aligned}$$

Therefore $\|T\| \leqslant M$, establishing the other inequality required to show (8.1). □

Notation: Given a pair of vectors $u, v \in \mathcal{H}$, $|u\rangle\langle v|$ stands for the operator $w \mapsto \langle v, w\rangle u$. In particular $P_u := |u\rangle\langle u|$ is the orthogonal projection onto the span of u.

Theorem 8.1.4 (Spectral Theorem for Compact Self-adjoint Operator). *Let $T \neq 0$ be a compact self-adjoint operator on $\mathcal{H}$. Then there exists a sequence $\{\lambda_n\}$ of real numbers and a sequence of mutually orthogonal vectors $\{e_n\}$ such that $|\lambda_n| \to 0, \|e_n\| = 1 \forall n$ and*

$$T = \sum \lambda_n |e_n\rangle\langle e_n|, \tag{8.3}$$

where the sum appearing in (8.3) is norm convergent. The expansion (8.3) is called a spectral resolution of T.

Proof. Let $T^{(0)} = T, \mathcal{H}^{(0)} = \mathcal{H}$. Now we will successively define

1. Hilbert spaces $\mathcal{H}^{(n)}$ for $n \geqslant 0$ such that $\mathcal{H}^{(n+1)} \subseteq \mathcal{H}^{(n)}$.
2. Compact self-adjoint operators $T^{(n)} : \mathcal{H}^{(n)} \to \mathcal{H}^{(n)}$.
3. Vectors $e_{n+1} \in \mathcal{H}^{(n)}$ orthogonal to $\mathcal{H}^{(n+1)}$ and scalars λ_{n+1} for $n \geqslant 0$.

This will be defined in a manner so that if $Q^{(n)}$ denotes the orthogonal projection onto $\mathcal{H}^{(n+1)}$ then

$$T^{(n+1)} = T^{(n)}Q^{(n)} = Q^{(n)}T^{(n)} \tag{8.4}$$

$$T^{(n)} = \lambda_{n+1}P_{e_{n+1}} + T^{(n+1)}, \text{ for } n \geqslant 0, \tag{8.5}$$

$$\|T^{(n+1)}\| \leqslant |\lambda_{n+1}|. \tag{8.6}$$

This is achieved through repeated applications of theorem (8.1.2). Assume that we have defined $(T^{(k)}, \mathcal{H}^{(k)})$ for $k \leqslant n$. If $T^{(n)} = 0$ then $T^{(n+1)} = 0, \lambda_{n+1} = 0, e_{n+1}$ an arbitrary unit vector in $\mathcal{H}^{(n)}$ and $\mathcal{H}^{(n+1)} = \mathcal{H}^{(n)} \cap \{e_{n+1}\}^{\perp}$,. Otherwise apply theorem (8.1.2) for the operator $T^{(n)}$.

$$(\lambda_{n+1}, e_{n+1}) = \begin{cases} (\Lambda_+(T^{(n)}), u_+(T^{(n)})), \text{ if } \Lambda_+(T^{(n)}) \geqslant -\Lambda_-(T^{(n)}) \\ (\Lambda_-(T^{(n)}), u_-(T^{(n)})) \text{ otherwise.} \end{cases}$$

Then $T^{(n)}e_{n+1} = \lambda_{n+1}e_{n+1}$ and consequently $\lambda_{n+1}P_{e_{n+1}} = T^{(n)}P_{e_{n+1}} = P_{e_{n+1}}T$. Let $Q^{(n)} = I_{\mathcal{H}^{(n)}} - P_{e_{n+1}}$ and $\mathcal{H}^{(n+1)}$ be the range of $Q^{(n)}$. If we take $T^{(n+1)} = T^{(n)}Q^{(n)}$ then all the conditions will be met. To see (8.6) observe that

$$\|T^{(n+1)}\| \leqslant \|T^{(n)}\| = |\lambda_{n+1}|, \text{ by lemma (8.1).}$$

Adding (8.5) for $0 \leqslant n \leqslant k$ we obtain,

$$T = \sum_{n=0}^{k} \lambda_{n+1}P_{e_{n+1}} + T^{(k+1)} \tag{8.7}$$

Since $\{e_n\}$ converges to zero weakly $|\lambda_n| = \|T(e_n)\|$ converges to zero. It follows from the inequality (8.6) that $\|T^{(n)}\|$ converges to zero. This proves (8.3). ☐

Definition 8.1.5. Let $T \in B(\mathcal{H})$, then λ is an eigenvalue of T with eigenvector $u \neq 0$ if $Tu = \lambda u$. The subspace $E_\lambda = \{u \in \mathcal{H} : Tu = \lambda u\}$ is called the eigenspace corresponding to the eigenvalue λ.

Corollary 8.1.6. Let $T \neq 0$ be a compact operator with a spectral resolution given by (8.3). Then $\lambda \neq 0$ is an eigenvalue iff λ equals one of the λ_n's. Also $E_\lambda = span\{e_n : \lambda_n = \lambda\}$.

Proof. Let A be the orthonormal set consisting of e_n's. Extend it to an orthonormal basis A'. Let $\lambda \neq 0$ be an eigenvalue with eigenvector u. Then by corollary (**??**) $u = \sum_n \langle e_n, u\rangle e_n + \sum_{\alpha \in A' \setminus A} \langle \alpha, u\rangle \alpha$. Therefore $Tu = \sum_n \lambda_n \langle e_n, u\rangle e_n$. On the other hand $\lambda u = \sum_n \lambda \langle e_n, u\rangle e_n + \sum_{\alpha \in A' \setminus A} \langle \alpha, u\rangle \alpha$. Using $Tu = \lambda u$ we obtain,

$$\langle \alpha, u\rangle = 0, \forall \alpha \in A' \setminus A \tag{8.8}$$

$$\lambda \langle e_n, u\rangle = \lambda_n \langle e_n, u\rangle, \forall n. \tag{8.9}$$

Equation (8.8) tells us u belongs to the closed linear span of e_n's. Hence there exists n such that $\langle e_n, u\rangle \neq 0$. Using equation (8.9) for that n we conclude $\lambda = \lambda_n$.

□

Corollary 8.1.7 (Singular Value Decomposition)**.** Let $T \neq 0$ be a compact operator. Then there exists countable orthonormal sets $\{e_n\}, \{f_n\}$ and a sequence of positive scalars $\{\lambda_n\}, \lambda_n \searrow 0$, such that

$$T = \sum_n \lambda_n |f_n\rangle\langle e_n| \tag{8.10}$$

where the sum is norm convergent.

Proof. Let $S = T^*T$. Then S is compact and nonzero because if $Tu \neq 0$ then $\langle u, Su\rangle = \|Tu\|^2 > 0$. Hence S is nonzero. □

Week 9

Harmonic Analysis on Compact Groups

9.1 Haar Measure

Let G be a topological group with a compact metrisable topology. This means G is a compact metrisable topological space and the group operations $m : G \times G \ni (g, h) \mapsto gh \in G$ and $G \ni g \mapsto g^{-1} \in G$ are continuous.

Definition 9.1.1. A probability measure λ on the Borel σ-algebra of G is said to be a left (right) Haar measure if $\lambda(gB) = \lambda(B)$ (respectively $\lambda(Bg) = \lambda(B)$) for allBorel sets B and for all $g \in G$. Clearly this implies for a left Haar measure λ, for every bounded Borel measurable function f we have $\int f(gh)d\lambda(h) = \int f(h)d\lambda(h)$. We have a similar conclusion for a right Haar measure.

We are interested in showing existence of a left Haar measure. Let us introduce few notations. Given $f \in C(G)$ and a signed measure μ we define functions as follows

$$(I \otimes \mu)(\Delta f)(g) = \int f(gh)d\mu(h)$$

$$(\mu \otimes I)(\Delta f)(h) = \int f(gh)d\mu(g).$$

In this notation λ is a left (right) Haar measure if $(I \otimes \lambda)(\Delta f)(g) = \int f(h)d\lambda(h)$ ($(I \otimes \lambda)(\Delta f)(h) = \int f(g)d\lambda(g)$) for all $f \in C(G)$. Another way to state the left Haar measure condition would be for all $f \in C(G)$, $(I \otimes \lambda)(\Delta f)$ is a constant function. So, to produce such a measure λ let us have an alternative description of constant functions.

Definition 9.1.2. A probability measure μ is said to be faithful if $\mu(U) > 0$ for all $U > 0$. If $\{x_n\}$ is a countable dense subset of G then we can take $\mu = \sum \frac{1}{2^n}\delta_{x_n}$, where δ_{x_n} is the Dirac mass at x_n.

Proposition 9.1.3. Let μ be a faithful measure on G. Then $f \in C(G)$ is a constant function iff $(I \otimes \mu)(\Delta f) = f$ iff $(\mu \otimes I)(\Delta f) = f$.

Proof. Only if parts are trivial because if f is the constant function $g \mapsto c$, then $(I \otimes \mu)(\Delta f) : g \mapsto c\mu(G)$. So, let us prove the if parts. I'll do that for the first one. So let $f \in C(G)$ be such that $(I \otimes \mu)(\Delta f) = f$. If f is not constant then there is some $\epsilon > 0$ and some non trivial open set U such that $f(g) < f(g_0) - \epsilon$ where $f(g_0) = \max_{g \in G} f(f)$. Using $(I \otimes \mu)(\Delta f) = f$ we get

$$\begin{aligned} f(g_0) &= (I \otimes \mu)(\Delta f)(g_0) \\ &= \int f(g_0 h) d\mu(h) \\ &= \int_{g_0^{-1}U} f(g_0 h) d\mu(h) + \int_{G \setminus g_0^{-1}U} f(g_0 h) d\mu(h) \\ &\leqslant (f(g_0) - \epsilon)\mu(g_0^{-1}U) + f(g_0)\mu(G \setminus g_0^{-1}U) \\ &= f(g_0) - \epsilon\mu(g_0^{-1}U) \\ &< f(g_0)! \end{aligned}$$

This contradiction shows f must be constant. □

In view of the previous proposition we have the following characterisation.

Proposition 9.1.4. Let μ be a faithful probability measure. Then λ is a left Haar measure iff for all $f \in C(G)$ we have $(I \otimes \mu)(\Delta(I \otimes \lambda)(\Delta f)) = (I \otimes \lambda)(\Delta f)$.

Exercise 9.1.5. State the corresponding proposition for a right Haar measure.

Proposition 9.1.6. Let μ, λ be probability measures on G and $f \in C(G)$, then

$$(I \otimes \mu)(\Delta(I \otimes \lambda)(\Delta f)) = (I \otimes (\mu \star \lambda)(\Delta f).$$

Proof.

$$\begin{aligned} (I \otimes \mu)(\Delta(I \otimes \lambda)(\Delta f))(g) &= \int (I \otimes \lambda)(\Delta f)(gh) d\mu(h) \\ &= \int\int f(ghh') d\lambda(h') d\mu(h) \\ &= \int f(gh) d(\mu \star \lambda)(h) \\ &= (I \otimes (\mu \star \lambda)(\Delta f)(g). \end{aligned}$$

□

In view of proposition (9.1.6) we can restate proposition (9.1.4) as follows.

Proposition 9.1.7. Let μ be a faithful probability measure on G. Then λ is a left Haar measure iff $\mu \star \lambda = \lambda$.

Theorem 9.1.8. *Let G be a topological group with a compact metrisable topology. Then G has a left Haar measure.*

Proof. Let μ be a faithful probability measure on G. We need to find a probability λ such that $\mu \star \lambda = \lambda$. Let $P(G) = \{\phi \in C(G)^* : \phi(1) = 1, \phi \text{ is positive}\}$. By Markov-Kakutani-Riesz representation theorem we can identify $P(G)$ with the collection of Borel probability measures on G. We will intentionally use the same symbol ϕ to denote the probability measure associated with the linear functional ϕ through this identification. Consequently if $\phi \in P(G)$ and $f \in C(G)$ then $\phi(f) = \int f d\phi$.

Claim: $P(G)$ is weak* closed: Let $\phi_\alpha \overset{w*}{\to} \phi$ be a convergent net from $P(G)$. We need to show $\phi \in P(G)$. That amounts to showing two things, first ϕ is a positive linear functional. That follows because if $f \in C(G)$ satisfies $f \geqslant 0$, then $\phi(f) = \lim \phi_\alpha(f) \geqslant 0$ since $\phi_\alpha(f) \geqslant 0$ for all α. Also $\phi(1) = \lim \phi_\alpha(1) = 1$.

Therefore by Banach Alaoglu theorem $P(G)$ is a compact subset in the weak* topology. Also $P(G)$ is easily to be convex. Since $P(G) \ni \lambda \mapsto \mu \star \lambda \in P(G)$ is an affine map, we will be done by Markov-Kakutani fixed point theorem once we show that $P(G) \ni \lambda \mapsto \mu \star \lambda \in P(G)$ is weak* continuous. Let $\lambda_\alpha \overset{w*}{\to} \lambda$. Then for all $f \in C(G)$ we have $\lim_\alpha \int f d\lambda_\alpha = \int f\lambda$. Therefore

$$\lim_\alpha \int f d(\mu \star \lambda_\alpha) = \lim_\alpha \int (\int f(gh) d\mu(g)) d\lambda_\alpha(h) = \int (\int f(gh) d\mu(g)) d\lambda(h) = \int f d(\mu \star \lambda)$$

In other words $(\mu \star \lambda_\alpha) \overset{w*}{\to} (\mu \star \lambda)$. Thus by Markov-Kakutani we obtain a probability measure λ such that $\mu \star \lambda = \lambda$. In view of proposition (9.1.7) this establishes existence of a Haar measure. □

Exercise 9.1.9. A left Haar measure is faithful.

Exercise 9.1.10. Let λ be a left Haar measure. Then show that $\lambda \star \lambda = \lambda$ and conclude that λ is a right Haar measure as well.

Exercise 9.1.11. Let λ_1, λ_2 be two left Haar probability measures. Show that $\lambda_1 = \lambda_2$.

9.2 Finite Dimensional Representations

Henceforth unless otherwise stated G will stand for a compact Hausdorff topological group. We have proved existence of Haar measure under the assumption of metrisability. Here we will assume it exists even without that assumption. Reference for that would be Functional Analysis books of Rudin and Conway. They give different proofs. We wish to understand strongly continuous unitary representations of G.

9.3 Schur Orthogonality

Definition 9.3.1. (1) A finite dimensional representation of G in a complex vector space V is a continuous homomorphism π from G to $GL(V)$, the space of invertible linear transformations of V. The vector space V is often referred as the representation space.
(2) A subspace W of V is called invariant if $\pi(g)(W) \subseteq (W)$ for all $g \in G$. It is called a reducing subspace if there exists another invariant subspace W' such that $V = W \oplus W'$. Clearly $\{0\}$ and V are invariant subspaces.
(3) A representation is called irreducible if it has no other invariant subspace.
(4) Let W be an invariant subspace, then the restriction of π to W denoted $\pi|_W$ is the representation $\pi|_W : G \to GL(W)$ given by $\pi|_W(g) := \pi(g)|_W$.
(5) A finite dimensional representation is called unitarizable if V can be endowed with an inner product such that each $\pi(G) \subseteq \mathcal{U}(V)$, the space of unitary operators.
(6) Given two representations $\pi_i : G \to G(V_i), i = 1, 2$, an intertwiner from π_1 to π_2 is a linear map $T : V_1 \to V_2$ such that $\pi_2(g)T = T\pi_1(g), \forall g \in G$. The representations are called equivalent if there exists an invertible intertwiner from π_1 to π_2.
(7) The direct sum of two representations $\pi_i : G \to GL(V_i), i = 1, 2$ is the representation $\pi : G \to GL(V)$ where $V = V_1 \oplus V_2$ and $\pi(g) = \pi_1(g) \oplus \pi_2(g) \in GL(V)$.
(8) A representation $\pi : G \to GL(V)$ is called completely reducible if there exists invariant subspaces $W_1, \cdots, W_n$ such that $V = \oplus_i^n W_i$ and each $\pi|_{W_i}$ is irreducible.

Proposition 9.3.2. Let $\pi : G \to GL(V)$ be a finite dimensional representation. Then it is unitarizable. In other words every finite dimensional representation is equivalent to a unitary representation.

Proof. Let n be the dimension of V. Then V is isomorphic with $\mathbb{C}^n$. Using any inner product on $\mathbb{C}^n$ we can define an inner product on V. Let $(\cdot, \cdot)$ be one such. Let $\langle u, v\rangle := \int_G (\pi(g)u, \pi(g)v) dg$, where dg denotes the Haar measure normalized so that measure of G is 1. Only thing we need to verify is $\langle u, u\rangle = 0$ implies $u = 0$. But that follows because $g \mapsto (\pi(g)u, \pi(g)u)$ is a nonnegative continuous function whose integral $\int_G (\pi(g)u, \pi(g)u) dg = \langle u, u\rangle$ vanishes. Therefore $(\pi(g)u, \pi(g)u) = 0$ for all $g \in G$. In particular taking $g = e$ we get $(u, u) = 0$. Since $(\cdot, \cdot)$ is an inner product we get $u = 0$. □

Corollary 9.3.3. Let $\pi : G \to GL(V)$ and W be an invariant subspace then it is reducing.

Proof. Fix an inner product on V such that π becomes a unitary representation. Let W' be the orthocomplement of W. Then $V = W \oplus W^\perp$. Only thing we need to show is that $W^\perp$ is invariant. Let $g \in G$ and $u \in W^\perp, w \in W$, then using the invariance of W we see that $\pi(g^{-1})w \in W$. Thus,

$$\langle \pi(g)u, w\rangle = \langle u, \pi(g)^* w\rangle = \langle u, \pi(g^{-1})w\rangle = 0.$$

This shows that $\pi(g)u \in W^\perp$, in other words $W^\perp$ is invariant. □

Corollary 9.3.4. Every finite dimensional representation is completely reducible.

Proof. The proof is by induction on the dimension of the representation space. If that is zero there is nothing to prove. Let us assume the that the result holds if the dimension of the representation space is less tan or equal to n. Now let $\pi : G \to GL(V)$ be a representation such that dimension of V is $n+1$. If the representation is ireducible there is nothing to prove, otherwise let W be an invariant subspace. We have just seen that then $W^{\perp}$ is also invariant. Clearly both W and $W^{\perp}$ have dimension less than n. By induction hypothesis there exists subspaces $W_1, \cdots, W_m$ of W and $W_{m+1}, \cdots, W_{m+k}$ of $W^{\perp}$ such that $W = \oplus_{i=1}^{m} W_i, W^{\perp} = \oplus_{i=m+1}^{m+k}$ and $\pi|_{W_i}$ is irreducible for $1 \leqslant i \leqslant m+k$. Then $V = \oplus_{i=1}^{m+k} W_i$ and π becomes completely reducible. □

Proposition 9.3.5 (Schur's lemma)**.** (1) Let $\phi_j : G \to GL(V_j), j = 1,2$ be finite dimensional irreducible representations of G. Suppose there exists a nonzero linear map $T : V_1 \to V_2$ such that $T\phi_1(g) = \phi_2(g)T, \forall g \in G$, then T is an isomorphism and consequently ϕ_1 and ϕ_2 becomes equivalent. In particular there is no nonzero intertwiner between inequivalent irreducible finite dimensional representations.
(2) Let $\phi : G \to GL(V)$ be an irreducible representation and $T : V \to V$ be a nonzero linear map such that $T\phi(g) = \phi(g)T, \forall g \in G$, then there exists a nonzero scalar $\lambda \in \mathbb{C}$ such that $T = \lambda I$.

Proof. (1) Let $W_1 = \ker(T)$ then this is an invariant subspace of V_1. Since ϕ_1 is irreducible there are two possibilities $W_1 = 0$ or V_1. The second possibility is ruled out because T is nonzero. Therefore T is one to one. Let $W_2 = \text{Image}(T)$, then this is an invariant subspace of V_2. As before there are two possibilities $W_2 = 0$ or V_2. The first possibility is ruled out because T is nonzero. Therefore T is onto.

(2) By part one we know that T is an isomorphism. Let λ be a nonzero eigenvalue of T. Then (i) $(T - \lambda.I)\phi(g) = \phi(g)(T - \lambda.I) \forall g \in G$, (ii) $\ker(T - \lambda.I)$ is a nonzero invariant subspace, hence must be whole of V. Therefore $T = \lambda.I$. □

Definition 9.3.6. Let $\phi : G \to GL(V)$ be a finite dimensional representaion. Let $\langle \cdot, \cdot \rangle$ be an inner product such that ϕ becomes unitary. Given a pair of vectors $v, v' \in V$, the continuous function $\theta_{\phi,v,v'}(g) = \langle v, \phi(g)v' \rangle$ is called a representation function. The continuous function $\chi_{\phi} : g \mapsto Tr(\phi(g))$ is called the character of the represetation ϕ. If we fix an orthonormal basis $u_1, \cdots, u_d$ of V then $\chi_{\phi}(g) = \sum_{j=1}^{d} \theta_{\phi,u_j,u_j}(g)$.

Proposition 9.3.7 (Schur Orthogonality Relations)**.** (1) Let $\phi_j : G \to GL(V_j), j = 1,2$ be two inequivalent irreducible representations. Let $v, v' \in V_1, w, w' \in V_2$. Then the $L_2(G)$ inner product of the associated representation functions $\theta_{\phi_1,v,v'}, \theta_{\phi_2,w,w'}$ vanishes. That is

$$\langle \theta_{\phi_1,v,v'}, \theta_{\phi_2,w,w'} \rangle = \int_G \overline{\langle v, \phi_1(g)v' \rangle} \langle w, \phi_2(g)w' \rangle dg = 0 \tag{9.1}$$

(2) Let $\phi : G \to GL(V)$ be an ireducible representation, then given four vectors $v, v', w, w' \in V$ the $L_2(G)$ inner product of the associated representation functions

$\theta_{\phi,v,v'}, \theta_{\phi,w,w'}$ is given by

$$\begin{aligned} \langle \theta_{\phi,v,v'}, \theta_{\phi,w,w'} \rangle &= \int_G \overline{\langle v, \phi(g)v' \rangle} \langle w, \phi(g)w' \rangle dg \\ &= \frac{1}{\dim V} \overline{\langle v, w \rangle} \langle v', w' \rangle. \end{aligned} \tag{9.2}$$

Proof. (1) Fix $v \in V_1, w \in V_2$. Consider the bilinear form

$$B : (v', w') \mapsto \int_G \overline{\langle v, \phi_1(g)v' \rangle} \langle w, \phi_2(g)w' \rangle dg.$$

Then

$$\begin{aligned} |B((v', w')| &\leqslant \int_G |\overline{\langle v, \phi_1(g)v' \rangle}| \cdot |\langle w, \phi_2(g)w' \rangle| dg \\ &\leqslant \int_G \|v\| \|\phi_1(g)v'\| \|w\| \|\phi_2(g)w'\| dg \\ &= \int_G \|v\| \|v'\| \|w\| \|w'\| dg, \text{ by unitarity of } \phi_1, \phi_2 \\ &= \|v\| \|v'\| \|w\| \|w'\| \end{aligned}$$

By remark (6.4.2) we obtain a bounded linear map $T : V_1 \to V_2$ such that $B(v', w') = \langle T(v'), w' \rangle$. Note that

$$\begin{aligned} \langle T\phi_1(h)(v'), \phi_2(h)(w') \rangle &= B(\phi_1(h)(v'), \phi_2(h)(w')) \\ &= \int_G \overline{\langle v, \phi_1(g)\phi_1(h)(v') \rangle} \langle w, \phi_2(g)\phi_2(h)(w') \rangle dg \\ &= \int_G \overline{\langle v, \phi_1(gh)v' \rangle} \langle w, \phi_2(gh)w' \rangle dg \\ &= B((v', w'), \\ &= \langle T(v'), w' \rangle. \end{aligned}$$

The fourth equality follows from the right invariance of Haar measure. It follows that T intertwines ϕ_1 and ϕ_2.

(1) If ϕ_1 and ϕ_2 are inequivalent then it follows from part (1) of proposition 9.3.5) that T must be zero. That is $B(v', w') = 0$. This proves (9.1).

(2) If $\phi_1 = \phi_2 = \phi$, then by part (2) of proposition 9.3.5) we conclude that $T = \lambda.I$ for some complex number λ. Thus,

$$\int_G \overline{\langle v, \phi(g)v' \rangle} \langle w, \phi(g)w' \rangle dg = \lambda \langle v', w' \rangle. \tag{9.3}$$

Let $d = \dim(V)$ and $u_1, \cdots, u_d$ be an orthonormal basis of V. In 9.3 we put $v' = w' = u_j$ and sum over j to obtain

$$\begin{aligned} d\lambda &= \int_G \sum_{j=1}^{d} \langle w, \phi(g)u_j\rangle\langle\phi(g)u_j, v\rangle dg \\ &= \int_G \sum_{j=1}^{d} \langle w, v\rangle dg [\text{ by } (\textbf{??})) \\ &= \langle w, v\rangle \end{aligned}$$

Thus $\lambda = \frac{\langle w,v\rangle}{d}$ and putting this in (9.3) we obtain (9.2). □

Corollary 9.3.8. Let ϕ, ϕ' be inequivalent irreducible reresentations. Then $\langle\chi_\phi, \chi_{\phi'}\rangle = 0$.

Definition 9.3.9. Let G be a compact hausdorff topological group and $\hat{G}_{fin}$ the space of equivalence classes of finite dimensional irreducible representations. Let d_ϕ denote the dimension of the representation space of ϕ. Let $L_2(\hat{G})$ be the completion of the pre-Hilbert space $\oplus_{\phi\in\hat{G}_{fin}} M(\mathbb{C}^{d_\phi})$ with respect to the inner product

$$\langle(T_\phi),(S_\phi)\rangle \sum_{\phi\in\hat{G}_{fin}} \frac{1}{d_\phi} \mathrm{Tr} T_\phi^* S_\phi.$$

Corollary 9.3.10. Let $\mathcal{F}_{\hat{G}} : L_2(\hat{G}) \longrightarrow L_2(G)$ be the map given by

$$E_{i,j,\phi} \mapsto \theta_{\phi,i,j}, \phi \in E_{i,j,\phi}, i = j = 1, \cdots, d_\phi$$

where $\{E_{i,j,\phi} : i,j = 1, \cdots, d_\phi\}$ is the canonical basis of $M(\mathbb{C}^{d_\phi})$. Then $\mathcal{F}_{\hat{G}}$ is an isometry.

Proof. Clearly $\{E_{i,j,\phi} : \phi \in \hat{G}_{fin}, i, j = 1, \cdots, d_\phi\}$ is an orthogonal basis of $L_2(\hat{G})$. It suffices to show that $\langle E_{i,j,\phi}, E_{k,l,\phi'}\rangle = \langle\theta_{\phi,i,j}, \theta_{\phi',k,l}, \forall\phi, \phi', i, j, k, l$. But that follows from the Schur orthogonality relations. □

9.4 The Banach-* Algebra of Square Integrable Functions

The space of square integrable functions form an algebra under convolution. In this section we will try to understand that.

Lemma 9.4.1. *Let G be a locally compact Hausdorff topological group and f be a compactly supported continuous function on G, then given $\epsilon > 0$ there exists a neighborhood U of identity $e \in G$ such that*

$$|f(g) - f(g')| < \epsilon, \textit{ for } g^{-1}g' \in U.$$

Proof. Let V be a relatively compact neighborhood of e. Then

$$W = V \cap V^{-1} = \{g \in G : g \in V, g^{-1} \in V\}$$

is a relatively compact neighborhood of e such that $g \in W$, implies $g^{-1} \in W$. Let $f' : G \times G \to \mathbb{C}$ be the continuous function given by $f'(g,h) = |f(g) - f(g.h)|$. Let C be the support of f. Then $D = C\overline{W}$ is a compact subset of G. Note that given any $g \in G$, $f'(g,e) = 0$, therefore there exists open neighborhoods A_g of g and $B_g \subseteq W$ of e such that for each $(g',h) \in (A_g \times B_g)$, $f'(g',h) = |f(g') - f(g'.h)| < \epsilon$. Let $g_1, \cdots, g_n$ be such that $A_{g_1}, \cdots, A_{g_n}$ covers D. Let $B = \cap B_{g_i}$ and $U = B \cap B^{-1}$. Let us take $g, g' \in G$ such that $g^{-1}g' \in U$. There are two possibilities.

1. If $g \in D$ then there is some i such that $g \in A_{g_i}$. Since $g^{-1}g' \in U \subseteq B_{g_i}$, there exists $h \in B_{g_i}$ such that $g' = g.h$ and
$$|f(g) - f(g')| = |f(g) - f(gh)| < \epsilon.$$
2. If $g \notin D$ then $g \notin C$. Then $g' = g.h \notin C$, because otherwise $g = g'h^{-1} \in CU \subseteq CW \subseteq D$. Therefore $|f(g) - f(g')| = 0$.

□

Proposition 9.4.2. Let G be a locally compact Hausdorff topological group and $1 \leqslant p < \infty$. For each $g \in G$ consider the linear map $L_g : L_p(G) \to L_p(G)$ given by $(L_g(f))(h) = f(g^{-1}h)$. Then

1. for all $f \in L_p(G)$, $\|L_g(f)\| = \|f\|$, in other words L_g is an isometry.
2. For all $g, h \in G, L_g L_h = L_{gh}$ and $L_e = Id$ where e is the identity of G.
3. For all $f \in L_p(G)$ the map $g \mapsto L_g(f)$ is a continuous map from G to $L_p(G)$.

Proof. (1) This follows from the left invariance of the Haar measure.

(2) This is also obvious.

(3) Note that by (1) and (2) $\|L_g(f) - L_h(f)\| = \|L_g(f - L_{g^{-1}}L_h(f))\| = \|f - L_{g^{-1}h}(f)\|$. Therefore it is enough to show that $g \mapsto L_g(f)$ is continuous at e. Let us firs assume that $f \in C_c(G)$. Applying lemma (9.4.1) for the compactly supported continuous function $g \mapsto f(g^{-1})$ we obtain an open set U such that $|f(g^{-1}) - f(g'^{-1}| < \epsilon$ provided $g^{-1}g' \in U$. Substituting $g^{-1} = h, g'^{-1} = h'$ we obtain

$$|f(h) - f(h')| \quad < \quad \epsilon, \text{ provided } hh'^{-1} \in U \tag{9.4}$$

Then given any $g \in U \cap U^{-1}$,

$$\|f - L_g(f)\|^p = \int_G |f(h) - f(g^{-1}h)|^p dh < 2\epsilon^p |supp(f)|$$

This shows that $g \mapsto L_g(f)$ is continuous. Now let us take an arbitrary $f \in L_p(G)$, then there exists $f' \in C_c(G)$ such that $\|f - f'\| < \epsilon/3$. There is an open neighborhood W of e such that $g \in W$ implies $\|f' - L_g(f')\| < \epsilon/3$. Then,

$$\|f - L_g(f)\| \leqslant \|f - f'\| + \|f' - L_g(f')\| + \|L_g(f') - L_g(f)\| < \epsilon.$$

□

Remark 9.4.3. If we take $p = 2$ then the resulting unitary representation is called the left regular representation and we have already encountered this in proposition (**??**). Instead of a left Haar measure if we had started with a right Haar measure λ' and considered the Hilbert space $L_2(G, \lambda')$ then the $R : G \to L_2(G, \lambda')$ given by $R_g \xi(h) = \xi(h.g)$ also gives a strongly continuous unitary representation. In case G is compact then a left invariant measure is also right invariant therefore both left and right regular representation acts on $L_2(G)$.

Lemma 9.4.4. *Let $f \in C(G)$, then $\int_G f(g^{-1})dg = \int_G f(g)dg$. If we define $\xi^* : g \mapsto \overline{\xi(g^{-1})}$ then $\xi \mapsto \xi^*$ extends to $L_2(G)$ as a conjugate linear involutive isometry. So, we have $\|\xi\|_2 = \|\xi^*\|_2, \forall \xi \in L_2(G)$.*

Proof. If we denote by λ the Haar measure then $\nu : E \mapsto \lambda(E^{-1})$ satisfies, $\nu(g.E) = \lambda(E^{-1}g^{-1}) = \lambda(E^{-1}) = \nu(E)$ for all $g \in G$ and every Borel set E. Also $\nu(G) = 1$. So, by uniqueness of the Haar measure we get $\nu = \lambda$. Thus,

$$\int_G f(g^{-1})d\lambda(g) = \int_G f(g)d\nu(g) = \int_G f(g)d\lambda(g).$$

Using this we see that

$$\|\xi\|_2^2 = \int_G |\xi(g)|^2 d\lambda(g) = \int_G |\xi(g^{-1})|^2 d\lambda(g) = \|\xi^*\|_2^2.$$

Clearly $(\xi^*)^* = \xi$ that is to say that the operation $\xi \mapsto \xi^*$ is involutive. □

Proposition 9.4.5. Let $f \in L_2(G)$ then $\mathcal{L}_f : \xi \mapsto f \star \xi$ defines a bounded linear map from $L_2(G)$ to $C(G)$. Image of the unit ball under this map is pointwise bounded and equicontinuous.

Proof. Let $\xi \in L_2(G)$. Since G is compact the Haar measure is finite and consequently $L_2(G) \subseteq L_1(G)$.

$$\begin{aligned} f \star \xi(h) &= \int_G f(g)\xi(g^{-1}h)dg \\ &= \int_G f(hg)\xi(g^{-1})dg, \quad [\text{ by a change of variable} \\ &= \langle \xi^*, L_{h^{-1}} f \rangle \end{aligned} \tag{9.5}$$

Using (9.5) along with Cauchy Schwarz inequality we get,

$$|f \star \xi(h) - f \star \xi(h')| \leqslant \|\xi\|_2 \|L_{h^{-1}} f - L_{h'^{-1}} f\|_2. \tag{9.6}$$

Since G is a topological group $g \mapsto g^{-1}$ is continuous. Therefore, by proposition (9.4.2,3) for any given f the map $g \mapsto L_{g^{-1}}f$ is continuous. That means given $\epsilon > 0$ there exists a neighborhood U of h' such that $h \in U$ implies $\|L_{h^{-1}}f - L_{h'^{-1}}f\|_2 < \epsilon$. If we combine this with (9.6) we see that

$$h \in U \Longrightarrow |f \star \xi(h) - f \star \xi(h')| \leqslant \epsilon \|\xi\|_2.$$

That shows the continuity of $f \star \xi$. In fact it also shows equicontinuity of the family $\{f \star \xi : \|\xi\|_2 \leqslant 1\}$. It remains to show the boundedness of $\mathcal{L}_f$. That follows from (9.5) once we note that

$$\|\mathcal{L}_f(\xi)\| \leqslant \sup_{h \in G} \|\xi\|_2 \|L_{h^{-1}}f\|_2 = \|\xi\|_2 \|f\|_2.$$

□

Corollary 9.4.6. Let $L_f : L_2(G) \to L_2(G)$ be the bounded linear map obtained by composing $\mathcal{L}_f : L_2(G) \to C(G)$ with the inclusion $C(G) \hookrightarrow L_2(G)$, then L_f is a compact operator.

Proof. Let $\{\xi_n\}$ be a sequence of unit vectors in $L_2(G)$. Then we know from proposition (9.4.5) that the Arzela-Ascoli theorem (**??**) applies. Thus there exists a Cauchy subsequence $\{\xi_{n_k}\}$ in $C(G)$. The inclusion $C(G) \hookrightarrow L_2(G)$ being continuous $\{\xi_{n_k}\}$ is Cauchy in $L_2(G)$. This shows that the image of the unit ball under L_f is relatively compact. □

Corollary 9.4.7. The Hilbert space $L_2(G)$ with convolution product is an involutive Banach algebra.

Proposition 9.4.8. The linear map L_f satisfies,

$$\langle L_f(\xi), \eta \rangle = \langle \xi, L_{f^*}(\eta) \rangle, \forall \xi, \eta \in L_2(G).$$

In particular L_f is self-adjoint provided $f = f^*$.

Proof. Let $f, \xi, \eta \in L_2(G)$, then

$$\begin{aligned}
\langle L_f(\xi), \eta \rangle &= \int_G \overline{f \star \xi(h)} \eta(h) dh \\
&= \int_G \int_G \overline{f(g)\xi(g^{-1}h)} \eta(h) dg dh \\
&= \int_G \int_G \overline{\xi(h')f(g)} \eta(gh') dh' dg \quad [g^{-1}h = h' \\
&= \int_G \int_G \overline{\xi(h')} f^*(g^{-1}) \eta(gh') dh' dg \\
&= \langle \xi, L_{f^*}(\eta) \rangle.
\end{aligned}$$

□

9.5 The Peter-Weyl Theory

Proposition 9.5.1. Let $f \in C(G)$ and $g \in G$, then $L_f R_g = R_g L_f$.

Proof. Let $\xi \in L_2(G)$, then

$$\begin{aligned}(L_f R_g \xi)(g') &= (f \star (R_g \xi))(g') \\ &= \int_G f(h)(R_g \xi)(h^{-1}g')dh \\ &= \int_G f(h)\xi(h^{-1}g'g)dh \\ &= f \star \xi(g'g) \\ &= (R_g L_f)(\xi)(g').\end{aligned}$$

This shows that $L_f R_g = R_g L_f$. □

Proposition 9.5.2. Let $f_1, \cdots, f_k \in C(G)$ then there exists a sequence of continuous functions δ_n such that $f_j \star \delta_n$ converges to f_j in $C(G)$ for $j = 1, \cdots, k$.

Proof. It is enough to show that given $\epsilon > 0$ there exists a continuous function δ such that

$$\forall g \in G, |f_j \star \delta(g) - f_j(g)| < \epsilon, \text{ for } 1 \leqslant j \leqslant k. \tag{9.7}$$

Using lemma (9.4.1) obtain an open set U such that

$$|f_j(g) - f_j(g')| < \epsilon, \text{ for } g^{-1}g' \in U, j = 1, \cdots, k. \tag{9.8}$$

Let $\delta \in C(G)$ be a compactly supported positive function such that

$$\overline{\{g \in G : \delta(g) \neq 0\}} \subseteq U, \text{ and } \int \delta(g)dg = 1.$$

$$\begin{aligned}|f_j \star \delta(g) - f_j(g)| &= |\int_G f_j(gh^{-1})\delta(h)dh - f_j(g)\int_G \delta(h)dh| \\ &\leqslant \int_G |f_j(gh^{-1}) - f_j(g)|\delta(h)dh \\ &= \int_U |f_j(gh^{-1}) - f_j(g)|\delta(h)dh \\ &\leqslant \int_U \epsilon\delta(h)dh, \qquad [\text{by}(9.8)] \\ &= \epsilon.\end{aligned}$$

□

Theorem 9.5.3 (Peter-Weyl). *Let G be a compact Hausdorff topological group and $f \in C(G)$ be a function with $f = f^*$. By corollary (9.4.6) and proposition (9.4.8) L_f is a compact self-adjoint operator. By the spectral theorem for compact operators Λ,the set of nonzero eigenvalues of L_f is a discrete subset of nonzero real numbers. For $\lambda \in \Lambda$, let $\mathcal{H}_\lambda = \{\xi \in L_2(G) : L_f(\xi) = \lambda\xi\}$ be the eigenspace for the eigenvalue λ. Then*

1. *Each $\mathcal{H}_\lambda$ for $\lambda \in \Lambda$ is a finite dimensional invariant subspace for $\mathcal{R}$ the right regular representation.*
2. *$\xi \in \mathcal{H}_\lambda, \delta \in C(G)$ implies $\xi \star \delta \in \mathcal{H}_\lambda$.*
3. *Let $\mathcal{R}_\lambda$ be the representation on $\mathcal{H}_\lambda$ obtained by restricting $\mathcal{R}$ to $\mathcal{H}_\lambda$. Then each element of $\mathcal{H}_\lambda$ is a representation function of $\mathcal{R}_\lambda$.*
4. *Given any $\epsilon > 0$ there exists a representation function h such that $\|f - h\|_2 < \epsilon$.*
5. *Given any $\epsilon > 0$ there exists a representation function h such that $\|f - h\|_\infty < \epsilon$.*

Proof. (1) Let $\xi \in \mathcal{H}_\lambda$. To show $\mathcal{R}(g)(\xi) \in \mathcal{H}_\lambda$ we must show $L_f(\mathcal{R}(g)(\xi)) = \lambda\mathcal{R}(g)(\xi)$. Since $L_f\mathcal{R}(g) = \mathcal{R}(g)L_f$ for all g we have

$$L_f(\mathcal{R}(g)(\xi)) = \mathcal{R}(g)(L_f\xi) = \mathcal{R}(g)(\lambda\xi) = \lambda\mathcal{R}(g)(\xi).$$

Thus $\mathcal{H}_\lambda$ is an invariant subspace for $\mathcal{R}$. By spectral theorem for compact selfadjoint operators each $\mathcal{H}_\lambda$ is finite dimensional.

(2) Since $L_f(\xi) = \lambda\xi$ we have $L_f(\xi \star \delta) = f \star (\xi \star \delta) = (f \star \xi) \star \delta = \lambda\xi \star \delta$. Therefore $\xi \star \delta \in \mathcal{H}_\lambda$.

(3) Let $\xi \in \mathcal{H}_\lambda$. Then $\xi \in C(G)$ because $\xi = \frac{1}{\lambda} f \star \xi$ and $f \star \xi \in C(G)$ by proposition (9.4.5). Therefore it mkes sense to evaluate ξ on an element of G. Let $\eta_1, \dots, \eta_{d_\lambda}$ be an orthonormal basis for $\mathcal{H}_\lambda$. Then $\mathcal{R}_\lambda(g)\xi = \sum_{j=1}^{d_\lambda} \langle \eta_j, \mathcal{R}_\lambda(g)\xi\rangle \eta_j$. Evaluating both sides on e, the identity element of G and using $\xi(g) = (\mathcal{R}(g)\xi)(e)$ we get $\xi(g) = \langle \sum_{j=1}^{d_\lambda} \overline{\eta_j(e)}\eta_j, \mathcal{R}(g)\xi\rangle =$ a representation function of $\mathcal{R}_\lambda$.

(4) Given $\epsilon > 0$, by proposition (9.5.2) obtain $\delta \in C(G)$ such that $\|f \star \delta - f\|_\infty < \epsilon/2$. Let $\{\lambda_i : i = 1, 2, \dots\}$ be an enumeration of elements of Λ in descending order of their absolute values. If we denote by P_i the orthogonal projection onto $\mathcal{H}_{\lambda_i}$, then the spectral theorem gives $L_f = \sum_i \lambda_i P_i$, where the right hand side is a norm convergent sum. Therefore

$$f \star \delta = L_f(\delta) = \lim_n \sum_{i=1}^{n} \lambda_i P_i(\delta). \tag{9.9}$$

By (3) each $P_i(\delta)$ is a representation function because it is in $\mathcal{H}_{\lambda_i}$. Also, representation functions form a linear space. Therefore, $\sum_{i=1}^{n} \lambda_i P_i(\delta)$ is a representation function. Thus by (9.9) we have shown $f \star \delta$ is an L_2 limit of representation functions. Therefore there is a representation function h such that $\|f \star \delta - h\|_2 < \epsilon/2$. Triangle inequality along with $\|\cdot\|_2 \leqslant \|\cdot\|_\infty$ gives

$$\|f - h\|_2 \leqslant \|f \star \delta - f\|_2 + \|f \star \delta - h\|_2 \leqslant \|f \star \delta - f\|_\infty + \epsilon/2 < \epsilon.$$

(5) Given $\epsilon > 0$, by proposition (9.5.2) obtain $\delta \in C(G)$ such that $\|f \star \delta - f\|_\infty < \epsilon/2$. By (4) obtain a representation function f' such that $\|f - f'\|_2 < \frac{\epsilon}{2\|\delta\|_2}$. Then for all $g' \in G$ we have

$$\begin{aligned}
|((f - f') \star \delta)(g')| &= |\int (f - f')(g'g)\delta(g^{-1})dg| \\
&\leqslant \sqrt{\int ((f - f')(g'g))^2 dg}\sqrt{\int (\delta(g^{-1}))^2 dg} \\
&= \sqrt{\int ((f - f')(g))^2 dg}\sqrt{\int (\delta(g))^2 dg} \\
&= \|f - f'\|_2 \|\delta\|_2 \\
&< \epsilon/2.
\end{aligned}$$

Therefore $\|((f - f') \star \delta)\|_\infty < \epsilon/2$. By triangle inequality

$$\|f' \star \delta - f\|_\infty \leqslant \|((f - f') \star \delta)\|_\infty + \|f \star \delta - f\|_\infty < \epsilon.$$

By (2) and (3) above $f' \star \delta$ is a representation function and we can take that as h. □

9.6 Fourier Series on Compact Groups

Let G be a comact, Hausdorff topological group and λ it's unique Haar measure normalized to have mass 1. We have already seen that representation functions are dense in $C(G)$. Now we seek to expand a function $f \in L_2(G)$ in an abstract Fourier series in terms of representation functions. So, let us fix notations. Recall we have denoted by $\widehat{G}$ the set of equivalence classes of finite dimensional irreducible unitary representations of G. Let us fix a representative from each class. Given a representation ϕ we will denote the representation space by $\mathcal{H}_\phi$. It is a finite dimensional Hilbert space, say of dimension d_ϕ. Fix an orthonormal basis $e_i, \ldots, e_{d_\phi}$ of $\mathcal{H}_\phi$. By Schur orthogonality relations $\{\sqrt{d_\phi}\theta_{\phi,i,j} : 1 \leqslant i, j \leqslant d_\phi\}$ is an orthonormal set. Let us record the following corollary of the Peter-Weyl theorem.

Corollary 9.6.1. The family $\cup_{\phi:[\phi]\in\widehat{G}}\{\sqrt{d_\phi}\theta_{\phi,i,j} : 1 \leqslant i, j \leqslant d_\phi\}$ is an orthonormal basis of $L_2(G)$.

Lemma 9.6.2. *Let $f \in L_2(G)$. Then the projection of f onto the span of $\{\sqrt{d_\phi}\theta_{\phi,i,j} : 1 \leqslant i, j \leqslant d_\phi\}$ is given by $d(f \star \chi_\phi)$.*

Proof. We need to calculate $\sum_{i,j}\langle\theta_{\phi,i,j},f\rangle\theta_{\phi,i,j}$.

$$\begin{aligned}
\sum_{i,j}\langle\theta_{\phi,i,j},f\rangle\theta_{\phi,i,j}(h) &= \sum_{i,j}\left(\int\overline{\langle e_i,\phi(g)e_j\rangle}f(g)d\lambda(g)\right)\langle e_i,\phi(h)e_j\rangle \\
&= \sum_{i,j}\int f(g)\langle\phi(g)e_j,e_i\rangle\langle e_i,\phi(h)e_j\rangle d\lambda(g) \\
&= \sum_{j}\int f(g)\langle\phi(g)e_j,\phi(h)e_j\rangle d\lambda(g) \\
&= \sum_{j}\int f(g)\langle e_j,\phi(g^{-1}h)e_j\rangle d\lambda(g) \quad [\text{since } \phi(g)^* = \phi(g^{-1})] \\
&= \int f(g)\chi_\phi(g^{-1}h)d\lambda(g) \\
&= f\star\chi_\phi(h).
\end{aligned}$$

□

Theorem 9.6.3. *Let* G *be a compact Hausdorff topological group and* $f\in L_2(G)$. *Then* $f=\sum_{\phi\in\widehat{G}}d_\phi f\star\chi_\phi$, *where the sum converges in* L_2 *norm.*

Proof. Corollary (9.6.1) and lemma (9.6.3) proves the result. □

Proposition 9.6.4. Let X be a compact Hausdorff space and $V: X\to\mathcal{L}(\mathcal{H})$ be a map such that $\forall v\in\mathcal{H}, x\mapsto V(x)(v)$ is continuous. Such maps will be referred as strongly continuous maps. Let μ be a probability measure on the Borel sigma-algebra of X. Then

(i). $\sup_{x\in X}\|V(x)\| = M < \infty$.

(ii). The map $B_V:\mathcal{H}\times\mathcal{H}\to\mathbb{C}$ given by $B(u,v)=\int\langle u,V(x)v\rangle d\mu(x)$ is conjugate linear in u and linear in v. Also $|B(u,v)|\leqslant\|u\|\|v\|M\mu(X)$.

(iii). There exists a bounded linear map to be denoted by $\int V(x)d\mu(x)$ such that $B(u,v)=\langle u,\left(\int V(x)d\mu(x)\right)(v)\rangle$.

Proof. (i) Strong continuity coupled with the continuity of nor implies that for all $v\in\mathcal{H}$ the map $x\mapsto\|V(x)(v)\|$ is continuous. Therefore $\forall v, \sup_{x\in X}\|V(x)(v)\|<\infty$, and by the Uniform Boundedness Principle we get the result.

(ii)Cauchy-Schwarz inequality and (i) implies

$$|\langle u,V(x)v\rangle|\leqslant\|u\|\|v\|M.$$

From this (ii) immediately follows.

(iii) Obvious. □

9.7 Projection Formulas

Let G be a compact, Hausdorff topological group and $V : G \to \mathcal{U}(\mathcal{H})$ a unitary representation which is also strongly continuous.

Week 10

Spectral Theory for Bounded Operators

10.1 Banach Algebras

Definition 10.1.1. A Banach algebra $\mathcal{A}$ is a Banach space along with an associative and distributive multiplication denoted $(a, b) \mapsto a.b$ such that $\|a.b\| \leqslant C\|a\|\|b\|$ for all $a, b \in \mathcal{A}$ for some positive C.

Remark 10.1.2. Let $\mathcal{A}$ be a Banach algebra. Then there exists an equivalent norm $\|.\|'$ on $\mathcal{A}$ such that for all $a, b \in \mathcal{A}$, $\|a.b\|' \leqslant \|a\|'\|b\|'$.

Proof. Suppose $\|a.b\| \leqslant C\|a\|\|b\| \forall a, b \in \mathcal{A}$
Case 1: $C < 1$, take $\|a\|' = \|a\|$
Case 2: $C > 1$, define $\|a\|' = C\|a\|$
In view of the above remark given any Banach algebra we will assume that the norm satisfies $\|a.b\| \leqslant \|a\|\|b\|$ for all $a, b \in \mathcal{A}$. □

Proposition 10.1.3. (1) Let $\mathcal{A}$ be a Banach algebra. Then $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ is a Banach algebra provided,

$$\begin{aligned} (x, \alpha).(y, \beta) &= (xy + \alpha y + \beta x, \alpha\beta) \\ \|(x, \alpha)\| &= \|x\| + |\alpha| \end{aligned}$$

(2) $x \mapsto (x, 0)$ gives an isometric embedding of $\mathcal{A}$ in $\widetilde{\mathcal{A}}$ as an ideal.
(3) $e = (0, 1)$ satisfies $(x, \alpha).e = e.(x, \alpha) = (x, \alpha)$ and $\|e\| = 1$.

Definition 10.1.4. A Banach algebra $\mathcal{A}$ with an element e such that $e.x = x.e = x \forall x \in \mathcal{A}$, $\|e\| = 1$ is called a unital Banach algebra.

Remark 10.1.5. The previous proposition says every Banach algebra can be isometrically embedded into a unital Banach algebra. Henceforth unless otherwise stated a Banach algebra means a unital Banach algebra.

Example 10.1.6. Let K be a compact Hausdorff space. $C(K)$ be the space of all continuous complex valued functions on K. For $f, g \in C(K)$, Define

$$\begin{aligned} (f+g)(p) &= f(p)+g(p) \\ (f.g)(p) &= f(p).g(p) \\ \|f\| &= \sup_{p\in K}|f(p)| \end{aligned}$$

$C(K)$ is a commutative Banach algebra.

Example 10.1.7. Let E be a Banach space. Then $\mathcal{L}(E)$, the space of all bounded linear maps from E to itself is a Banach algebra under operator norm.

Example 10.1.8. Let K be a compact subset of $\mathbb{C}$ or $\mathbb{C}^n$ with nonempty interior. Then $\mathcal{A} = \{f \in C(K) : f|_{\text{interior of } K}$ is holomorphic$\}$ is a Banach algebra.

Proposition 10.1.9. Let G be a locally compact group. Let μ be a Haar measure on G. Recall that μ satisfies

$$\int f(gh)d\mu(h) = \int f(h)d\mu(h).$$

Then $\mathcal{A} = L_1(G, \mu)$ is a Banach algebra with multiplication defined by

$$(f_1 \star f_2) = \int f_1(g)f_2(g^{-1}h)d\mu(g).$$

Proof. (1) $f_1 \star f_2 \in L_1$:

$$\begin{aligned} \int |f_1 \star f_2(h)d\mu(h) &\leqslant \int\int |f_1(g)||f_2(g^{-1}h|d\mu(g)d\mu(h) \\ &= \int |f_1(g)|d\mu(g) \int |f_2(h)|d\mu(h) \\ &= \|f_1\|_1\|f_2\|_1 \end{aligned}$$

Therefore we have proved

$$\begin{aligned} f_1 \star f_2 &\in L_1(G) \text{and} \\ \|f_1 \star f_2\|_1 &\leqslant \|f_1\|_1\|f_2\|_1 \end{aligned}$$

(2) $(f_1 \star f_2) \star f_3 = f_1 \star (f_2 \star f_3)$:

$$\begin{aligned} (f_1 \star f_2) \star f_3(u) &= \int (f_1 \star f_2)(v)f_3(v^{-1}udv \\ &= \int\int f_1(w)f_2(w^{-1}v)f_3(v^{-1}u)dwdv \\ &= \int\int f_1(w)f_2(v)f_3(v^{-1}w^{-1}u)dwdv \\ &= f_1 \star (f_2 \star f_3)(u) \end{aligned}$$

□

Example 10.1.10. Let $C^1[0,1]$ be the space of once continuously differentiable functions. Define $\|f\| = \sup_{x\in[0,1]}|f(x)| + \sup_{x\in[0,1]}|f'(x)|$. Then under pointwise multiplication $C^1[0,1]$ is a Banach algebra.

Example 10.1.11. Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a norm closed subalgebra. D a densely defined closed operator. D need not be bounded. Let

$$\begin{aligned}\mathcal{A}^{(1)} = \{a \in \mathcal{A} \quad : \quad & a(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D) \text{ and } \forall \xi \in \mathrm{Dom}(D) \\ & \exists C > 0 \text{ such that } \|[D,a]\xi\| < C\|\xi\|\}\end{aligned}$$

$\mathcal{A}^{(1)}$ is a Banach algebra with the norm $\|a\|^{(1)} = \|a\| + \|[D,a]\|$.

Proof. Suppose $\{a_n\}$ is a Cauchy sequence with respect to $\|\cdot\|^{(1)}$. Then $a_n \to a$, and $[D, a_n] \to b$ For $\xi \in \mathrm{Dom}(D)$ we have

$$\begin{aligned} Da_n\xi &\to b\xi + aD\xi \\ a_n\xi &\to a\xi \end{aligned}$$

Since D is closed

$$\begin{aligned} (i) a\xi &\in \mathrm{Dom}(D) \\ (ii) Da\xi &= aD\xi + b\xi \\ \text{Therefore } [D,a] &= b \\ \text{So, } a &\in \mathcal{A}^{(1)} \end{aligned}$$

Therefore $\mathcal{A}^{(1)}$ is complete. For $a, b \in \mathcal{A}^{(1)}$,

$$\begin{aligned} \|ab\|^{(1)} &= \|ab\| + \|[D,ab]\| \\ &\leqslant \|a\|\|b\| + \|[D,a]b + a[D,b]\| \\ &\leqslant \|a\|^{(1)}\|b\|^{(1)}. \end{aligned}$$

□

Theorem 10.1.12. Assume that $\mathcal{A}$ is a Banach space as well as a complex algebra with a unit element $e \neq 0$, in which multiplication is both left and right continuous. Then there is a norm on $\mathcal{A}$ which induces the same topology as the given one and makes $\mathcal{A}$ a Banach algebra.

Proof. Define $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{A})$ by, $\pi(x)(z) = xz$. Clearly $\pi(x)$ is linear. It is continuous because multiplication is given to be right continuous. $\|x\| = \|xe\| = \|\pi(x)(e)\| \leqslant \|\pi(x)\|\|e\|$, So π is one to one. We also have $\|\pi(x)\pi(y)\| \leqslant \|\pi(x)\|\|\pi(y)\|$, $\|\pi(e)\| = 1$. So $\pi(\mathcal{A})$ is a Banach algebra provided it is complete. For that it is enough to show that $\pi(\mathcal{A})$ is closed. For that suppose $\pi(x_n) \to T$ in $\mathcal{L}(\mathcal{A})$. Then $x_n = \pi(x_n)(e) \to T(e) = x$.

$$T(y) = \lim \pi(x_n)(y) = \lim x_n y = xy = \pi(x)(y)$$

by continuity of left multiplication. So $T = \pi(x)$. □

Definition 10.1.13. A linear map $\phi : \mathcal{A} \to \mathcal{B}$ is called a homomorphism if

$$\begin{aligned} \phi(xy) &= \phi(x)\phi(y), \quad \forall x, y \in \mathcal{A} \\ \|\phi(x)\| &\leqslant \|x\| \quad \forall x \in \mathcal{A}. \end{aligned}$$

A nonzero homomorphism into the complex numbers is called a complex homomorphism

Proposition 10.1.14. If ϕ is a complex homomorphism on a Banach algebra $\mathcal{A}$ then $\phi(e) = 1$ and $\phi(x) \neq 0$ for all invertible $x \in \mathcal{A}$.

Proof. For some $y \in \mathcal{A}$, $\phi(y) \neq 0$, $\phi(y) = \phi(y)\phi(e)$ gives $\phi(e) = 1$.
$\phi(x)\phi(x^{-1}) = \phi(e) = 1$ gives $\phi(x) \neq 0$. □

10.2 Spectrum

Proposition 10.2.1. Let $x \in \mathcal{A}$ with $\|x\| < 1$ then $(I - x)$ is invertible.

Proof. The series $\sum_{n=0}^{\infty} x^n$ converges and is the inverse of $(I - x)$. □

Corollary 10.2.2. Let $G(\mathcal{A})$ be the set of invertible elements of a Banach algebra $\mathcal{A}$. Then $G(\mathcal{A})$ is an open subset of $\mathcal{A}$.

Proof. Let $x \in G(\mathcal{A})$. For $y \in \mathcal{A}$ with $\|y\| < \frac{1}{\|x\|^{-1}}$, $(x - y) = x^{-1}(I - x^{-1}y)$ is invertible by the previous proposition because $\|x^{-1}y\| \leqslant \|x^{-1}\|\|y\| < 1$. □

Definition 10.2.3. Let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$. Then the spectrum of x is defined as $\{\lambda \in \mathbb{C} : (\lambda - x) \text{ is not invertible}\}$. It is denoted by $\sigma_{\mathcal{A}}(x)$. We often drop the subscript $\mathcal{A}$. For a nonunital Banach algebra $\mathcal{A}$ the spectrum of an element x is defined as $\sigma_{\widetilde{\mathcal{A}}}(x)$ where $\widetilde{\mathcal{A}}$ is the unitization defined before.

Definition 10.2.4. The spectral radius $\rho(x)$ of $x \in \mathcal{A}$ is defined as

$$\rho(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

Definition 10.2.5 (The resolvent set)**.** The complement of spectrum of $x \in \mathcal{A}$ is called the resolvent of x and is also denoted by $\rho(x)$. We have also used same notation for spectral radius. Both notations are standard. You have to make out from the context.

Definition 10.2.6 (The resolvent function)**.** Let $x \in \mathcal{A}$. Then for $\lambda \in \rho(x)$, the function $\lambda \mapsto \mathcal{R}_\lambda(x) = (\lambda 1_{\mathcal{A}} - x)^{-1}$ is called the resolvent function.

Proposition 10.2.7. Let x be an element of a Banach algebra $\mathcal{A}$. Then $\sigma(x)$ is a nonempty closed and bounded subset of $\mathbb{C}$.

Proof. $\sigma(x)$ is closed:
Enough to show that it's complement is open. Suppose λ is such that $(\lambda - x)$ is invertible. Then by the proof of the previous corollary the ball of radius $\frac{1}{\|\lambda - x\|^{-1}}$ around λ is contained in $\sigma(x)^c$. Hence $\sigma(x)^c$ is open.
$\sigma(x)$ is bounded:
If λ is such that $|\lambda| > \|x\|$ then $(\lambda - x) = \lambda(I - \frac{x}{\lambda})$ is invertible. Hence $\sigma(x)$ is contained in the ball of radius $\|x\|$.
$\sigma(x)$ is nonempty:
If possible let $\sigma(x)$ be empty. Then $f(\lambda) = (\lambda - x)^{-1}$ is a holomorphic function defined on the entire plane. For $\lambda > \|x\|$, we have

$$\begin{aligned} f(\lambda) &= \lambda^{-1}\left(I - \frac{x}{\lambda}\right)^{-1} \\ &= \lambda^{-1}\sum_{n=0}^{\infty} x^n \lambda^{-n}, \text{since} \|\frac{x}{\lambda}\| < 1 \\ \text{So}, \|f(\lambda)\| &\leqslant |\lambda|^{-1}\frac{|\lambda|}{|\lambda| - \|x\|} \\ &\leqslant \frac{1}{|\lambda| - \|x\|} \end{aligned}$$

Hence f is a bounded entire function. Therefore it must be constant. From the previous estimates we see that $\lim_{\lambda\to\infty} f(\lambda) = 0$. Hence f is the constant function 0. But 0 is not invertible so we get a contradiction. □

Theorem 10.2.8 (Gelfand-Mazur)**.** *Let $\mathcal{A}$ be a Banach algebra such that every nonzero element is invertible then $\mathcal{A} \cong \mathbb{C}$.*

Proof. Suppose $\lambda_1 \neq \lambda_2 \in \sigma(x)$, then $(x - \lambda_1) = 0 = (x - \lambda_2)$. Hence, $\sigma(x)$ consists of a single point say $\lambda(x)$, and $x = \lambda(x)I$. $x \mapsto \lambda(x)$ gives an isomorphism between $\mathcal{A}$ and $\mathbb{C}$. □

Lemma 10.2.9. Let R be a commutative ring over $\mathbb{C}$. Then ab is invertible iff a and b are invertible.

Proof. Suppose $c = (ab)^{-1} = (ba)^{-1}$. Then $a^{-1} = bc$ because, (i) $abc = 1$ (ii) $bca = abc = 1$, the first equality uses commutativity. □

Proposition 10.2.10. Let p be a polynomial. Then for any $x \in \mathcal{A}$, $\sigma(p(x)) = p(\sigma(x))$.

Proof. Let $\lambda \in \mathbb{C}$.

$$\begin{aligned} p(z) - \lambda &= c\prod(z-\lambda_i), \quad \text{for some } c \neq 0, \lambda_1, \cdots, \lambda_n \in \mathbb{C} \\ p(x) - \lambda &= c\prod(x-\lambda_i) \end{aligned}$$

$\sigma(p(x)) \subseteq p(\sigma(x))$:

$$\begin{aligned} \lambda \in \sigma(p(x)) &\implies \lambda_i \in \sigma(x) \text{ for some } i \\ &\implies \lambda \in p(\sigma(x)) \text{ since } \lambda = p(\lambda_i) \end{aligned}$$

$p(\sigma(x)) \subseteq \sigma(p(x))$:

$$\begin{aligned} \lambda \in p(\sigma(x)) &\implies \lambda = p(\mu) \text{ for some } \mu \in \sigma(x) \\ &\implies p(x) - \lambda = (x-\mu)q(x) \text{ for some polynomial } q \\ &\implies \lambda \in \sigma(p(x)) \text{ by the lemma above} \end{aligned}$$

□

Proposition 10.2.11. Let x be an element of the Banach algebra $\mathcal{A}$. Then the spectral radius satisfies $\rho(x) = \mathrm{Lim}\|x^n\|^{\frac{1}{n}} = \inf\|x^n\|^{1/n}$

Proof. By the previous lemma $\rho(x^n) = \rho(x)^n, \forall n \geqslant 1$, also $\rho(x) \leqslant \|x\|$. So, $\rho(x)^n = \rho(x^n) \leqslant \|x\|^n$ implying $\rho(x) \leqslant \inf\|x^n\|^{1/n} \leqslant \underline{\mathrm{Lim}}\|x^n\|^{\frac{1}{n}}$. To complete the proof it suffices to show $\overline{\mathrm{Lim}}\|x^n\|^{\frac{1}{n}} \leqslant \rho(x)$. Let ϕ be a continuous linear functional on $\mathcal{A}$. Then the resolvent

$$f(\lambda) = (\lambda - x)^{-1} = \lambda^{-1}(1-\lambda^{-1}x)^{-1}$$

is holomorphic outside the disk of radius $\rho(x)$. So, $g(\lambda) = \lambda(1-\lambda x)^{-1}$ is analytic inside the disk of radius $\frac{1}{\rho(x)}$. For $|\lambda| < \|x\|$ we have the power series expansion $g(\lambda) = \sum \lambda^{n+1}x^n$. The function $\lambda \mapsto (\phi \circ g)(\lambda)$ is holomorphic in the disk of radius $\frac{1}{\rho(x)}$. Hence it's Taylor series $\sum \phi(x^n)\lambda^{n+1}$ converges in this disk. Thus

$$|\phi(\lambda^n x^n)| \to 0 \qquad \text{if } |\lambda|\rho(x) < 1.$$

For each fixed ϕ and λ we have some constant $C(\lambda, \phi)$ such that

$$\mathrm{Sup}_n|\phi(\lambda^n x^n)| < C(\lambda, \phi).$$

For each $|\lambda| < \frac{1}{\rho(x)}$ consider the family of linear functionals on $\mathcal{A}^*$ given by $T_n : \phi \mapsto \phi(\lambda^n x^n)$. We know

$$\mathrm{Sup}_n|T_n(\phi)| < C(\lambda, \phi).$$

By the uniform boundedness principle we get

$$\sup_n \|T_n\| < C(\lambda) \text{ for some constant } C(\lambda).$$

Clearly $\|T_n\| = \|\lambda^n x^n\|$, so

$$\begin{aligned} & \|x^n\| < C(\lambda)|\lambda|^{-n} \text{ for } |\lambda| < \frac{1}{\rho(x)} \\ \implies & \|x^n\|^{1/n} < C(\lambda)^{1/n}|\lambda|^{-1} \text{ for } |\lambda| < \frac{1}{\rho(x)} \\ \implies & \overline{\text{Lim}}\|x^n\|^{1/n} < |\lambda|^{-1} \text{ for } |\lambda| < \frac{1}{\rho(x)} \\ \implies & \overline{\text{Lim}}\|x^n\|^{1/n} \leqslant \rho(x) \end{aligned}$$

□

10.3 Holomorphic Function Calculus

Let $x \in \mathcal{A}$ and Ω an open neighborhood of $\sigma(x)$. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Let Γ be a contour in Ω surrounding $\sigma(x)$. Then define

$$f(x) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\lambda)}{\lambda - x}d\lambda.$$

Note that on the resolvent set $\lambda \mapsto \frac{f(\lambda)}{\lambda - x}$ is holomorphic. Hence the integral is well defined and does not depend on Γ.

Proposition 10.3.1. The mapping $f \mapsto f(x)$ is a homomorphism from the algebra of functions holomorphic in a neighborhood of $\sigma(x)$ to $\mathcal{A}$. Moreover if $f_k : \Omega \to \mathbb{C}$ is given by $f_k(z) = z^k$ then $f_k(x) = x^k$.

Proof. Let f and g be holomorphic functions defines in a neighborhood of $\sigma(x)$. Let C_1, C_2 be curves surrounding $\sigma(x)$ such that C_2 lies inside C_1. Then,

$$\begin{aligned} f(x)g(x) &= \left(\frac{1}{2\pi i}\int_{C_1}\frac{f(\lambda)}{\lambda - x}d\lambda\right)\left(\frac{1}{2\pi i}\int_{C_2}\frac{g(\mu)}{\mu - x}d\mu\right) \\ &= -\frac{1}{4\pi^2}\int_{C_1}\int_{C_2} f(\lambda)g(\mu)(\lambda - x)^{-1}(\mu - x)^{-1}d\lambda d\mu \\ &= -\frac{1}{4\pi^2}\int_{C_1}\int_{C_2}\frac{f(\lambda)g(\mu)}{\lambda - \mu}\left(\frac{1}{\mu - x} - \frac{1}{\lambda - x}\right)d\lambda d\mu \\ &= -\frac{1}{4\pi^2}\int_{C_1}\int_{C_2}\frac{f(\lambda)g(\mu)}{\lambda - \mu}\frac{1}{\mu - x}d\lambda d\mu \\ &= \frac{1}{2\pi i}\int_{C_2}\left(\frac{1}{2\pi i}\int_{C_1}\frac{f(\lambda)}{\lambda - \mu}d\lambda\right)\frac{g(\mu)}{\mu - x}d\mu \\ &= \frac{1}{2\pi i}\int_{C_2}\frac{f(\mu)g(\mu)}{\mu - x}d\mu \\ &= (f.g)(x) \end{aligned}$$

Here the fourth equality follows from $\frac{1}{4\pi^2}\int_{C_1}\left(\int_{C_2}\frac{g(\mu)}{\lambda-\mu}d\mu\right)\frac{f(\lambda)}{\lambda-x} = 0$. This is so because $\frac{g(\mu)}{\lambda-\mu}$ is holomorphic inside C_2 if λ lies in C_1.
To show $f_k(x) = x^k$, let $C = \{\lambda : |\lambda| = \|x\| + \epsilon\}$ for some $\epsilon > 0$.

$$\begin{aligned} f(x) &= \frac{1}{2\pi i}\int_C \sum_{n=0}^{\infty} \frac{x^n\lambda^k}{\lambda^{n+1}} d\lambda \\ &= x^k \end{aligned}$$

□

Proposition 10.3.2 (Spectral Mapping Theorem)**.** Let $x \in \mathcal{A}$ and f be a holomorphic function in a neighborhood of $\sigma(x)$. Then $\sigma(f(x)) = f(\sigma(x))$. Moreover if g is holomorphic in a neighborhood of $f(\sigma(x))$, then we have $(g \circ f)(x) = g(f(x))$.

Proof. Let $y = f(x)$ and $\mu \not\in f(\sigma(x))$. Since $f(\sigma(x))$ is compact $\exists U$ open with $\overline{U}$ compact $f(\sigma(x)) \subseteq U \subseteq \overline{U} \subseteq \{\mu\}^c$. On $f^{-1}(U)$ define a holomorphic function $h(\lambda) = \frac{1}{f(\lambda)-\mu}$. Put $z = h(x)$, then by the previous proposition $(y-\mu)z = (f(x)-\mu)h(x) = (h(f-\mu))(x) = 1$, hence $\mu \notin \sigma(f(x))$. On the other hand if $\mu \in f(\sigma(x))$, then $\mu = f(\lambda_0)$ for some $\lambda_0 \in \sigma(x)$. Then there exists holomorphic h around $\sigma(x)$ such that

$$\begin{aligned} f(\lambda) - \mu &= (\lambda - \lambda_0)h(\lambda) \\ \text{so, } (y - \mu) &= (x - \lambda_0)h(x) \end{aligned}$$

Since $(x - \lambda_0)$ is not invertible neither is $(y - \mu)$. Thus $\mu \in \sigma(f(x))$. Now choose simple closed curves C_1 and C_2 in such a way that C_1 encloses $f(\sigma(x)$ and is in the domain of g and C_2 encloses the inverse image of C_1 under f and is contained in the domain of f.

$$\begin{aligned} (g\circ f)(x) &= \frac{1}{2\pi i}\int_{C_1}\frac{(g\circ f)(\lambda)}{\lambda - x}d\lambda \\ &= \frac{-1}{4\pi^2}\int_{C_1}\left(\int_{C_2}\frac{g(\mu)}{\mu - f(\lambda)}d\mu\right)\frac{1}{\lambda - x}d\lambda \\ &= \frac{-1}{4\pi^2}\int_{C_2} g(\mu)\int_{C_1}\left(\frac{1}{\mu - f(\lambda)}\right)\left(\frac{1}{\lambda - x}\right)d\lambda \\ &= \frac{1}{2\pi i}\int_{C_2}\frac{g(\mu)}{\mu - f(x)}d\mu \\ &= g(f(x)) \end{aligned}$$

□

10.4 Abelian Banach Algebras

In this section unless otherwise stated we are dealing with a not necessarilly unital commutative Banach algebra $\mathcal{A}$.

Definition 10.4.1. An ideal $\mathfrak{m}$ of $\mathcal{A}$ is called regular the quotient ring $\mathcal{A}/\mathfrak{m}$ is unital. In other words if there exists $e \in \mathcal{A}$ such that $\forall x \in \mathcal{A}, \quad ex - x \in \mathfrak{m}$.

Proposition 10.4.2. Let $\mathfrak{m}$ be a proper regular ideal of $\mathcal{A}$. If e is an identity modulo $\mathfrak{m}$, then we have

$$\inf\{\|e - x\| : x \in \mathfrak{m} \geqslant 1.$$

Proof. Suppose $\|e-x\| < 1$ for some $x \in \mathfrak{m}$. Then the power series $y = \sum_{n=1}^{\infty} (e-x)^n$ converges. Since $(e-x)y = \sum_{n\geqslant 2} (e-x)^n$, we have

$$\begin{aligned} y &= (e-x) + (e-x)y \\ &= ey - xy + e - x. \end{aligned}$$

Hence $e = y - ey + xy + x \in \mathfrak{m}$. For any $a \in \mathcal{A}$, $a = ea + (a - ea) \in \mathfrak{m}$. Thus $\mathfrak{m} = \mathcal{A}$, a contradiction! □

Corollary 10.4.3. The closure of any regular proper ideal of an abelian Banach algebra $\mathcal{A}$ is proper and regular. In particular any maximal regular ideal is closed.

Proposition 10.4.4. Any proper regular ideal is contained in a maximal regular ideal.

Proof. Let e be an identity modulo $\mathfrak{m}$. Then any ideal containing $\mathfrak{m}$ is regular. Now apply Zorn's lemma to ideals containing $\mathfrak{m}$ and not containing e. □

Proposition 10.4.5. Let $\mathfrak{m}$ be a closed ideal of a possibly noncommutative Banach algebra $\mathcal{A}$. The quotient algebra $\mathcal{A}/\mathfrak{m}$ is a Banach algebra.

Proof. Let $\pi : \mathcal{A} \to \mathcal{A}/\mathfrak{m}$ be the quotient map. From the definition of the quotient norm it follows that $\|\pi(x)\| = \inf\{\|x + m\| : m \in \mathfrak{m}\}$. Given $\epsilon > 0$ get m, n from $\mathfrak{m}$ such that $\|x + m\| \leqslant \|\pi(x)\| + \epsilon$, $\|y + n\| \leqslant \|\pi(y)\| + \epsilon$.

$$\begin{aligned} \|\pi(x)\pi(y)\| &= \|\pi(xy)\| = \|\pi((x+m)(y+n))\| \\ &\leqslant \|(x+m)(y+n)\| \\ &\leqslant (\|\pi(x)\| + \epsilon)(\|\pi(y)\| + \epsilon) \end{aligned}$$

Since ϵ is arbitrary $\|\pi(x)\pi(y)\| \leqslant \|\pi(x)\|\|\pi(y)\|$. □

Proposition 10.4.6. Let $\mathcal{A}$ be a unital Banach algebra. If an element $x \in \mathcal{A}$ is not invertible then x is contained in some maximal ideal.

Proof. $\mathcal{A}x$ is a proper regular ideal. Hence there exists a maximal ideal containing this. □

Proposition 10.4.7. Let $\phi : \mathcal{A} \to \mathbb{C}$ be a nonzero complex homomorphism. Then $\phi^{-1}(0)$ is a regular maximal ideal. $\phi \mapsto \phi^{-1}(0)$ gives a bijection between nonzero complex homomorphisms and regular maximal ideals of $\mathcal{A}$.

Proof. Since $\mathcal{A}/\mathrm{Ker}(\phi)$ is isomorphic with a field $ker(\phi)$ is a regular maximal ideal. To show that the correspondence is bijective observe that for a regular maximal ideal $\mathfrak{m}$, $\mathcal{A}/\mathfrak{m}$ is a Banach algebra with every nonzero element being invertible. This is so because otherwise by the above proposition we will get a contradiction to the maximality of $\mathfrak{m}$. Now by the Gelfand-Mazur theorem $\mathcal{A}/\mathfrak{m} \cong \mathbb{C}$. Hence $\mathfrak{m} = ker(\phi)$ where, $\phi : \mathcal{A} \to \mathcal{A}/\mathfrak{m}$ is the quotient map. □

Proposition 10.4.8. Let ω be a nonzero complex homomorphism of $\mathcal{A}$. Then $\|\omega\| \leqslant 1$.

Proof. We have,

$$|\omega(x)| = |\omega(x^n)|^{1/n} \leqslant \|\omega\|^{1/n}\|x^n\|^{1/n}$$

Now taking limit as n goes to infinity we get $|\omega(x)| \leqslant \rho(x) \leqslant \|x\|$. Therefore $\|\omega\| \leqslant 1$. □

Proposition 10.4.9. (i) Let $\Omega(\mathcal{A})$ be the set of all nonzero complex homomorphisms. Then under weak* topology $\Omega(\mathcal{A})$ is a locally compact Hausdorff space.
(ii) If $\mathcal{A}$ is unital, then $\Omega(\mathcal{A})$ is compact.
(iii) For $x \in \mathcal{A}$, $\widehat{x} : \Omega(\mathcal{A}) \to \mathbb{C}$ defined by $\widehat{x}(\omega) = \omega(x)$ gives a homomorphism $\mathcal{F} : \mathcal{A} \to C_0(\Omega(\mathcal{A}))$, called Gelfand transform.
(iv) For $\mathcal{A}$ unital we have $\sigma(x) = \{\widehat{x}(\omega) : \omega \in \Omega(\mathcal{A})\}$. For $\mathcal{A}$ nonunital $\sigma(x) = \{\widehat{x}(\omega) : \omega \in \Omega(\mathcal{A})\} \cup \{0\}$.
(v) $\|\widehat{x}\| = \rho(x)$.

Proof. (i) Let $\Omega' = \Omega \cup \{0\}$ and ω_i be a convergent net in Ω'. Suppose $\omega_i \to \omega$ in weak* topology. Then $\omega(xy) = lim\omega_i(xy) = lim\omega_i(x)\omega_i(y) = \omega(x)\omega(y)$. Therefore ω is a homomorphism. It may be the zero homomorphism. Being a weak* closed subset of the unit ball of $\mathcal{A}^*$ Ω' is compact. Clearly $\{0\}$ is closed. Hence $\Omega(\mathcal{A})$ is locally compact. Suppose $\omega_1 \neq \omega_2 \in \Omega(\mathcal{A})$. Then there exists $x \in \mathcal{A}$ such that $|\omega_1(x) - \omega_2(x)| > \epsilon$ for some $\epsilon > 0$. Note that $\{\omega : |\omega_1(x) - \omega(x)| < \epsilon/3\}$ and $\{\omega : |\omega_2(x) - \omega(x)| < \epsilon/3\}$ are disjoint neighborhoods of ω_1 and ω_2. Hence Ω' is Hausdorff.
(ii) If $\mathcal{A}$ is unital then $\{0\}$ is an isolated point in Ω' because for any other $\omega \in \Omega' \omega(1) = 1$. Hence Ω is compact.
(iii) $\widehat{x} \in C_0(\Omega(\mathcal{A}))$ because for any $\epsilon \geqslant 0, \{\omega : |\widehat{x}(\omega)| \geqslant \epsilon\}$ is compact. Clearly $\mathcal{F}$ is a homomorphism.
(iv) Case 1 $\mathcal{A}$ Unital : If $\lambda \in \sigma(x)$ then $(x - \lambda)$ is not invertible. Hence there exists $\omega \in \Omega(\mathcal{A})$ such that $\omega(x - \lambda) = 0$ or equivalently $\lambda = \widehat{x}(\omega)$. So, $\lambda \in$ Range of $\widehat{x}$. Conversely suppose $\lambda = \widehat{x}(\omega) = \omega(x)$, then $\omega(x - \lambda) = 0$. Hence $\lambda \in \sigma(x)$.
(v) Follows from (iv). Note that this implies that the Gelfand transform is contractive. □

Definition 10.4.10. Let $\mathcal{A}$ be a commutative Banach algebra then $\Omega(\mathcal{A})$ is called the space of characters of $\mathcal{A}$ or the spectrum of $\mathcal{A}$.

10.5 Characters of $L_1(G)$

Let G be a locally compact abelian group and μ, a left invariant Haar measure. Then we have seen the abelian Banach algebra $L_1(G, \mu)$. We wish to identify its space of characters.

Theorem 10.5.1. *Let ω be a character of $L_1(G)$, that is to say that it is a nonzero homomorphism from $L_1(G)$ to the complex numbers. Then there is a continuous homomorphism $\phi : G \to \mathbb{T}$ such that $\omega(f) = \int_G f(g)\phi(g)dg$.*

Proof. In particular ω is a bounded linear functional on $L_1(G)$, hence there exists $\phi \in L_\infty(G)$ such that $\omega(f) = \int_G f(g)\phi(g)dg$.

$$\begin{aligned} \omega(f_1 \star f2) &= \int_G (f_1 \star f_2)(h)\phi(h)dh \\ &= \int_G \int_G f_1(g)f_2(g^{-1}h)\phi(h)dgdh \\ &= \int_G f_1(g)(\int_G L_g(f_2)(h)\phi(h)dh)dg \\ &= \int_G f_1(g)\omega(L_g(f_2))dg \end{aligned}$$

On the other hand

$$\begin{aligned} \omega(f_1 \star f_2) &= \omega(f_1)\omega(f_2) \\ &= \omega(f_2)\int_G f_1(g)\phi(g)dg. \end{aligned}$$

Therefore ,

$$\int_G \omega(f_2)f_1(g)\phi(g)dg = \int_G f_1(g)\omega(L_g(f_2))dg, \forall f_1, f_2 \in L_1(G). \quad (10.1)$$

Since ω is a nonzero homomorphism there exists f_2 such that $\omega(f_2) \neq 0$. It follows from (10.1) that

$$\phi(g) = \frac{\omega(L_g(f_2))}{\omega(f_2)}, a.e \quad (10.2)$$

Note that ϕ is determined upto a set of measure zero. However Part (3) of proposition (9.4.2) along with (10.2)shows that ϕ is almost everywhere equal to a continuous function namely $\frac{\omega(L_g(f_2))}{\omega(f_2)}$ and we will take this representative. In particular

$\phi(e) = 1$. To see that ϕ is multiplicative note that given arbitrary $f_1, f_2 \in L_1(G)$,

$$\begin{aligned}
0 &= \omega(f_1 \star f_2) - \omega(f_1)\omega(f_2) \\
&= \int_G \int_G f_1(g) f_2(g^{-1}h)\phi(h) dg dh - (\int_G f_1(g)\phi(g)dg)(\int_G f_2(h)\phi(h)dh) \\
&= \int_G \int_G f_1(g) f_2(g^{-1}h)\phi(gg^{-1}h) dh dg - \int_G \int_G f_1(g) f_2(h)\phi(g)\phi(h) dg dh \\
&= \int_G \int_G f_1(g) f_2(h')\phi(gh') dh' dg - \int_G \int_G f_1(g) f_2(h)\phi(g)\phi(h) dg dh, \\
&\qquad [\text{substituiting } g^{-1}h = h',] \\
&= \int_G \int_G f_1(g) f_2(h)(\phi(gh) - \phi(g)\phi(h)) dg dh.
\end{aligned}$$

Since ϕ is continuous this shows that ϕ is a homomorphism, that is

$$\phi(gh) = \phi(g)\phi(h), \forall g, h \in G.$$

It remains to show that $|\phi(g)| = 1, \forall g \in G$. Suppose there exists $\alpha > 1$ such that the open set $A_\alpha = \{g \in G : |\phi(g)| > \alpha\}$ is non-empty. Fix a compact subset K of A_α of positive measure. Define

$$f(g) = \begin{cases} \overline{\frac{\phi(g)}{|\phi(g)|}} & \text{if } g \in K, \\ 0, & \text{otherwise} \end{cases}.$$

Then $\|f\|_1 = |K|$, where $|K|$ denotes Haar measure of K. Let $\tilde{f} = \frac{f}{\|f\|_1}$. By proposition (10.4.8) we have

$$\begin{aligned}
1 \geqslant \|\omega\|.\|\tilde{f}\| \geqslant |\omega(\tilde{f})| &= \int_K \overline{\frac{\phi(g)}{|\phi(g)|}} \frac{\phi(g)}{|K|} dg \\
&= \int_K \frac{|\phi(g)|}{|K|} dg > \alpha > 1!
\end{aligned}$$

This contradiction shows that A_α must be empty. That is $|\phi(g)| \leqslant 1$ for all $g \in G$. Similarly considering $\phi(g)^{-1}$ we conclude that $|\phi(g)| \geqslant 1$ for all $g \in G$. Thus we get range of ϕ is contained in $\{z \in \mathbb{C} : |z| = 1\}$.

□

10.6 C*-algebras

Definition 10.6.1. A Banach algebra A is called involutive if there exists a map $* : A \to A$ such that $a \mapsto a^*$ satisfies

$$\begin{aligned}
(a + \lambda b)^* &= a^* + \bar{\lambda} b^*, \\
(ab)^* &= b^* a^*, \\
(a^*)^* &= a, \\
\|x^*\| &= \|x\|.
\end{aligned}$$

An involutive Banach algebra A is called a C^*-algebra if $\|x^*x\| = \|x\|^2$ for all $x \in A$.

$x \in A$ is called hermitian or selfadjoint if $x = x^*$, normal if $xx^* = x^*x$, unitary if $x^*x = xx^* = I$, projection if $x = x^* = x^2$.

Proposition 10.6.2. Let A be a C^*-algebra. If $x \in A$ is normal then $\|x\| = \rho(x)$.

Proof. $\|x^2\|^2 = \|(x^2)^*x^2\| = \|(x^*x)^2\| = \|x^*x\|^2 = \|x\|^4$. Therefore we have, $\|x^2\| = \|x\|^2$, implying $\|x^{2^n}\| = \|x\|^{2^n}$. So $\rho(x) = \|x\|$. □

Proposition 10.6.3. Let A be a unital C^*-algebra.

1. $\sigma(u) \subseteq \{\lambda : |\lambda\| = 1\}$ for all unitary u.
2. $\sigma(h) \subseteq \mathbb{R}$ for all hermitian h.

Proof. (1) $\|u\|^2 = \|u^*u\| = \|I\| = 1 \Longrightarrow \|u\| = 1$. Therefore $\sigma(u)$ is contained in the unit disc. Also u is invertible with $u^{-1} = u^*$. Therefore 0 does not belong to $\sigma(u)$. Therefore by the spectral mapping theorem we have $\sigma(u^{-1}) \subseteq \{z \in \mathbb{C} : |z| \geqslant 1\}$. On the otherhand $\|u^{-1}\| = \|u^*\| = 1$, hence $\sigma(u^{-1}) \subseteq \{z \in \mathbb{C} : |z| \leqslant 1\}$. Therefore $\sigma(u^{-1}) \subseteq \{z \in \mathbb{C} : |z| = 1\}$. Now by the spectral mapping theorem we are done.

(2)$u = e^{ih}$ is a unitary. Hence by the spectral mapping theorem we have $e^{i\sigma(h)} \subseteq \{z \in \mathbb{C} : |z| = 1\}$. The only way this can happen is $\sigma(h) \subseteq \mathbb{R}$. □

Theorem 10.6.4. *Let $\mathcal{A}$ be an abelian C^*-algebra. If Ω is the spectrum of A, then the Gelfand transformation is an isometric isomorphism of A onto $C_0(\Omega)$, preserving the $*$-operation.*

Proof. We know $\|\widehat{x}\| = \rho(x)$. On the other hand since $\mathcal{A}$ is abelian every element is normal. So, $\|x\| = \rho(x)$. Therefore the Gelfand transform $x \mapsto \widehat{x}$ is isometric. Take $\omega \in \Omega$, for $h \in \mathcal{A}_h$, $\omega(h) \in \sigma(h) \subseteq \mathbb{R}$. x can be expressed as $x = h + ik$, with $h, k \in \mathcal{A}_h$. $\omega(x^*) = \omega(h - ik) = \omega(h) - i\omega(k) = \overline{\omega(x)}$. Hence $x \mapsto \widehat{x}$ preserves *-operation.
Let $\mathcal{F} : \mathcal{A} \to C_0(\Omega)$, $\mathcal{F}(x) = \widehat{x}$, then $\mathcal{F}(\mathcal{A})$ separates points because if $\omega_1 \neq \omega_2 \in \Omega$, then there exists $x \in \mathcal{A}$ such that $\omega_1(x) \neq \omega_2(x)$. Hence $\widehat{x}(\omega_1) \neq \widehat{x}(\omega_2)$. By the Stone-Weirstrass theorem$\mathcal{F}\mathcal{A} = C_0(\Omega)$. □

Proposition 10.6.5. Let Ω be a locally compact Hausdorff space and $\mathcal{A} = C_0(\Omega)$. The map $\omega \in \Omega \mapsto \widehat{\omega} \in \Omega(\mathcal{A})$ given by $\widehat{\omega}(x) = x(\omega)$ is a homeomorphism of Ω onto $\Omega(\mathcal{A})$.

Proof. Let us assume Ω to be compact. Then $\Omega(\mathcal{A})$ is compact and $\omega \mapsto \widehat{\omega}$ is continuous because if $\omega_\alpha \to \omega$ then $x(\omega_\alpha) \to x(\omega) \forall x \in \mathcal{A}$, or equivalently $\widehat{\omega}(x) \to \omega(x)$. $\omega \mapsto \widehat{\omega}$ is one to one: Suppose $\omega_1 \neq \omega_2$, then by Tietze extension theorem $\exists f$ such that $f(\omega_1) = 0$ and $f(\omega_2) = 1$. $\widehat{\omega_1}(f) \neq \widehat{\omega_2}(f)$.

$\omega \mapsto \widehat{\omega}$ is onto: Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{A}$. Then $\exists\omega$ such that $\mathfrak{m} = \{x : x(\omega) = 0\}$. Let ϕ be the homomorphism corresponding to $\mathfrak{m}$,i.e., $\phi : \mathcal{A} \to \mathcal{A}/\mathfrak{m}, \phi(x) = x(\omega)$. Then $\widehat{\omega} = \phi$. So $\omega \mapsto \widehat{\omega}$ is a bijective map between compact Hausdorff spaces. Hence it is a homeomorphism.

If Ω is locally compact and not compact then argue through one point compactification. □

Proposition 10.6.6. Let $\mathcal{B} \subseteq \mathcal{A}$ be a C^*-subalgebra of a unital C^*-algebra containing the identity. Then $\forall x \in \mathcal{B} \sigma_{\mathcal{B}}(x) = \sigma_{\mathcal{A}}(x)$.

Proof. Case 1: Let x be self adjoint.
Clearly $\sigma_{\mathcal{A}}(x) \subseteq \sigma_{\mathcal{B}}(x)$. Suppose $\lambda \in \mathbb{R} \setminus \sigma_{\mathcal{A}}(x)$ we want to show $\lambda \notin \sigma_{\mathcal{B}}(x)$. For $\epsilon > 0, \lambda_\epsilon = \lambda + i\epsilon \notin \sigma_{\mathcal{B}}(x)$, hence $(x - \lambda_\epsilon)^{-1} \in \mathcal{B}$. Using continuity of inverse in $G(\mathcal{A})$, we get $(x-\lambda_\epsilon)^{-1} \to (x-\lambda)^{-1}$ in $G(\mathcal{A})$. Since $\mathcal{B}$ is closed, $(x-\lambda)^{-1} \in \mathcal{B}$, hence $\lambda \notin \sigma_{\mathcal{B}}(x)$.
Case 2: If $x \in \mathcal{B}$ is invertible in $\mathcal{A}$ then x^*x is invertible in $\mathcal{A}$ and so in $\mathcal{B}$ (By the previous case). Hence x is left invertible in $\mathcal{B}$. Similarly using xx^* x is right invertible in $\mathcal{B}$. Hence x is invertible in $\mathcal{B}$. So, $\lambda \notin \sigma_{\mathcal{A}}(x)$ iff $(x - \lambda)$ is invertible in $\mathcal{A}$ iff $(x - \lambda)$ is invertible in $\mathcal{B}$ iff $\lambda \notin \sigma_{\mathcal{B}}(x)$. □

Proposition 10.6.7. Let $\mathcal{A}$ be a unital C^*-algebra. If $x \in \mathcal{A}$ is normal then there exists a unique isomorphism $\phi : C(\sigma(x)) \to C^*(x)$, the C^*-algebra generated by x and 1 such that $\phi(i) = 1, \phi(\iota) = x$ where $\iota : \sigma(x) \to \mathbb{C}$ is the function $\iota(\lambda) = \lambda$.

Proof. Let $\mathcal{B} = C^*(x)$ and $\mathcal{P}$ = polynomials in x and x^*. $\mathcal{P}$ is dense in $\mathcal{B}$. Let Ω = space of all complex homomorphisms from $\mathcal{B}$ to $\mathbb{C}$. Define $\psi : \Omega \to \sigma(x)$ by $\psi(\eta) = \eta(x)$.
$\psi(\eta) \in \sigma(x)$: $\eta(x - \eta(x)) = 0$, hence $x - \eta(x)$ is not invertible.
ψ is continuous: Suppose $\eta_\alpha \to \eta$ in weak*, then $\eta_\alpha(x) \to \eta(x)$ in $\mathbb{C}$.
ψ is one to one: Suppose η_1 and η_2 are two homomorphisms such that $\eta_1(x) = \eta_2(x)$, then $\eta_1|_{\mathcal{P}} = \eta_2|_{\mathcal{P}}$. Since $\mathcal{P}$ is dense in $\mathcal{B}$, $\eta_1 = \eta_2$.
ψ is onto: Suppose $\lambda \in \sigma(x)$, then $\exists\eta$ such that $\lambda = \eta(x)$. $\psi(\eta) = \lambda$.
ψ is a bijective continuous map between compact Hausdorff spaces and hence a homeomorphism. ψ induces an isomorphism between $C(\Omega)$ and $C(\sigma(x))$. This isomorphism composed with the inverse of the Gelfand transform gives the required isomorphism. In other words $\phi(f) = \mathcal{F}^{-1}(f \circ \psi)$ is the isomorphism. □

Definition 10.6.8 (Continuous Function Calculus)**.** Let $x \in \mathcal{A}$ be a normal element. Let f be a complex valued continuous function on $\sigma(x)$. Then $\phi(f)$ with ϕ as in the previous proposition is denoted by $f(x)$.

Proposition 10.6.9. Let $\mathcal{A}$ be a unital C^*-algebra. Then every element of $\mathcal{A}$ is a linear combination of 4 unitary elements.

Proof. Let $x \in \mathcal{A}$ be selfadjoint and $\|x\| \leqslant 1$. $u = x + i(1 - x^2)^{1/2}$ is a unitary and $x = \frac{1}{2}(u + u^*)$. □

Proposition 10.6.10. Let $K \subseteq \mathbb{C}$ be compact. $A_K = \{x \in \mathcal{A} | x \text{ is normal and } \sigma(x) \subseteq K\}$. If $f : K \to \mathbb{C}$ is continuous then $x \in A_L \mapsto f(x) \in \mathcal{A}$ is continuous.

Proof. By Stone-Weirstrass there ixists a polynomial $p(z, \bar{z})$ such that

$$\mathrm{Sup}_{z \in K} |p(z, \bar{z}) - f(z)| < \epsilon$$

There exists a constant M such that $\|x\| < M$ for $x \in A_K$. Also, since p is a polynomial $\exists \delta > 0$ such that

$$\|p(x, x^*) - p(y, y^*)\| < \epsilon \text{ if } \|x - y\| < \delta, \|x\|, \|y\| < M.$$

Now if $x, y \in A_K$ and $\|x - y\| < \delta$, then $\|f(x) - f(y)\| \leqslant \|f(x) - p(x, x^*)\| + \|p(x, x^*) - p(y, y^*)\| + \|f(y) - p(y, y^*\| < 3\epsilon$. □

Theorem 10.6.11 (Not Done in Class)**.** *For a selfadjoint element x in a C^* algebra $\mathcal{A}$, the following are equivalent.*
(i) $\sigma(x) \in [0, \infty)$.
(ii) $x = y^* y$ *for some* $y \in \mathcal{A}$.
(iii) $x = h^2$ *for some* $h \in \mathcal{A}$.
The set of all selfadjoint elements satisfying any of the above is a closed convex cone P *in* $\mathcal{A}$ *with* $P \mathcal{A} p(-P) = \{0\}$