Functional Analysis

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Preface

This is the third run of my course on Functional Analysis at ISI. First time I taught this to M.Stat students in the first semester of 2022-23. Second run of the course was during the second semester of 2022-23 to a batch of M.Math students. This time Jan 2025-April 2025, it will be to a batch of M.Math students. I am planning a change in the presentation and content. Not sure whether I'll be able to type my notes. Let's see.

Rules of The Game

We do experiments to learn about things. There are people who recognise their significance and are explicit about it. This explicitness is also at the core of mathematical methods. So, this semester we will do several of them.

The Checklist

You are attending classes for several years and if not all, I am sure many of you use a checklist to assist you in your studies. May be we can do that officially. Here is a checklist. You should use this list regularly. I wrote down whatever came to my mind and as we go along we will append this.

Checklist

- Did I introduce any concept? If so then what is that concept? You must pay attention to concepts being introduced.
- Was the concept introduced out of thin air or were their attempts to motivate the introduction?
- Did you feel motivated? Note that what constitutes motivation etc. are subjective issues but that does not mean we can't talk about them and introduce objectivity. If you are not motivated did you raise any objection? You can still do it. Only condition is, you must be able to write down your objection clearly. At the end of the day, the subject of Mathematics is about one and only one thing clarity. Clarity

of thought expressed through precise linguistic means characterises this subject.

- Was the concept illustrated through examples? Do you want more? This has a catch though because you have to answer when do you consider two examples to be different. Of course I'll assist you there.
- Do you think the purpose behind the introduction of the concept has been achieved? If not, you should come back to this question in future and check again.
- Can you summarise the material covered in this class? Or in this topic/subtopic?

Later on we may and I am sure we will append this checklist. We will communicate on this matter through other channels like WA.

Taking Notes

It is better to take your own notes.

Weightage

Classtest 10(5+5), Assignent+Notes+Viva 10, Midsem 30, End semester 50.

Chapter 1

Getting Ready

Functional Analysis is the study of linear algebra coupled with topological considerations. Even though this could be an almost accurate and possibly shortest description of the subject, in reality it did not start with such considerations. It is not that one fine morning someone thought let's see what happens if we club topology and linear algebra together. No, meaningful subject starts in that way. It came up in our endeavour to answer very natural questions. As we go along I hope to indicate more impressive reasons behind topologizing linear algebra. However, for the time being we will remain content with this naive motivation.

1.1 Nets

Since we wish to topologize linear algebra we begin by asking how can we specify a topology. This could be done, for example, by specifying the class of closed sets or equivalently by specifying the operation of taking closures. We have seen this while introducing topologies associated with metric spaces. The following proposition shows that the concept of sequential convergence allows us to define the notion of closure of a set in a metric space.

Proposition 1.1.1. Let (X, d) be a metric space and $A \subseteq X$. Then closure of A is given by

$$\overline{A} := \{\lim x_n : \{x_n\} \subseteq A, \text{ is a sequence }\}$$

We also know that such a proposition does not hold in a topological space unless that is first countable. Is there a natural generalisation of sequential convergence so that there is an analog of this proposition in the setting of topological spaces? This was answered by E.H. Moore and Herman L. Smith in 1922, in the article "A General Theory of Limits" spanning pages 102-121, published in the 2nd issue of the 44th volume of the journal American Journal of Mathematics. Instead of writing such long sentences we could have written "this was answered by, E. H. Moore ; H. L. Smith, A General Theory of Limits, American Journal of Mathematics, 1922, 44 (2), 102-12" and in future will write this way. Here is what Moore and Smith did. They described a natural generalisation of a sequence called nets and that allowed them to obtain closure of a set in a general topological space through what they called convergence of nets. Lets see that.

Definition 1.1.2 (Preorder)**.** A binary relation $\preceq$, on a set Λ is called reflexive if $\lambda \preceq \lambda, \forall \lambda \in \Lambda$. The relation $\preceq$ is said to be transitive if given any three elements $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ satisfying $\lambda_1 \preceq \lambda_2 \preceq \lambda_3$ we have $\lambda_1 \preceq \lambda_3$. A reflexive, transitive binary relation is called a preorder. A preordered set $(\Lambda, \preceq)$ is a set Λ, equipped with a preorder $\preceq$.

Definition 1.1.3 (Directed Set)**.** A preordered set $(\Lambda, \preceq)$ is called directed if

$$\forall \lambda_1, \lambda_2 \in \Lambda, \exists \lambda \in \Lambda, \lambda_1 \preceq \lambda, \lambda_2 \preceq \lambda.$$

A preorder with this property is called a direction.

Example 1.1.4.

1. $(\mathbb{N}, \preceq)$ with $n \preceq m$ iff $n \leqslant m$ is a directed set.
2. $((0, \infty), \preceq)$ with $y \preceq x$ iff $y \leqslant x$ is a directed set.
3. $((0, 1), \preceq)$ with $y \preceq x$ iff $x \leqslant y$ is a directed set.

Example 1.1.5. Let X be a topological space and $x \in X$. Let N_x be the set of neighbourhoods of x. Consider the relation $W \preceq V$ if $V \subseteq W$. Then $(N_x, \preceq)$ is a directed set.

Exercise 1.1.6. Are the examples in 1.1.4 special cases of example 1.1.5?

[Lecture Notes of P.S.Chakraborty]

Definition 1.1.7. Let $(\Lambda_i, \preceq_i)_{i\in I}$ be a family of directed sets. Then $\Lambda := \prod_i \Lambda_i$ becomes a directed set with $(a_i)_{i\in I} \preceq (b_i)_{i\in I}$ if $a_i \preceq_i b_i, \forall i$. The directed set $(\Lambda, \preceq)$ is called the product directed set of the family $(\Lambda_i, \preceq_i)_{i\in I}$. Unless otherwise specified we will always endow products of directed sets with this direction.

Definition 1.1.8 (Net)**.** Let X be a set and $(\Lambda, \preceq)$ be a directed set. A net $\{x_\lambda\}_{\lambda\in\Lambda}$ in X with index set Λ is a map $x : \Lambda \ni \lambda \mapsto x_\lambda \in X$. When the index set is understood we drop it from the notation and just say, $\{x_\lambda\} \subseteq X$ is a net.

Example 1.1.9. Every sequence defines a net with the index set $(\mathbb{N}, \leqslant)$.

Definition 1.1.10 (Convergence of nets)**.** A net $\{x_\lambda\}_{\lambda\in\Lambda}$ in a topological space X is said to converge to some point $x \in X$ if for each neighbourhood V of x, there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in V$ whenever $\lambda_0 \preceq \lambda$. In that case we say x is a limit of the net $\{x_\lambda\}$ and write $x_\lambda \to x$.

Proposition 1.1.11. A topological space X is Hausdorff iff every net in X converges to at most one point.

Proof. If part: Suppose X is not Hausdorff. Then $exists x, y \in X$ such that $\forall U \in N_x, \forall V \in N_y, U \cap V \neq \emptyset$. For each $(U, V) \in N_x \times N_y, x_{U,V} \in U \cap V$. Then the net $\{x_{U,V}\}_{(U,V)\in N_x\times N_y}$ converges to both x and y.

Only if part is left as an exercise. □

Exercise 1.1.12. Prove only if part of the proposition 1.1.11.

Now we will show that the analog of proposition 1.1.1 holds.

Proposition 1.1.13. Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ iff x is a limit of a net $\{x_\alpha\} \subseteq A$.

Proof. Let $x \in \overline{A}$. If $V \in N_x$, then $V \cap A \neq \emptyset$, so there exists $x_V \in V \cap A$. Then the net $\{x_V\}_{V\in N_x}$ converges to x. Conversely if $x_\alpha \to x$ and $\{x_\alpha\} \subseteq A$ then $x \in \overline{A}$. □

Definition 1.1.14 (Subnet)**.** A net $\{y_\beta\}_{\beta\in B}$ is a subnet of a net $\{x_\alpha\}_{\alpha\in A}$ if there is an order preserving function $\phi : B \to A$ such that (i) for all $\alpha_0 \in A$, there exists $\beta_0 \in B$ such that $\phi(\beta_0) \succeq \alpha_0$, in other words $\phi(B)$ is a cofinal subset of A and (ii) for all $b \in B, y_b = x_{\phi(b)}$.

Remark 1.1.15. In the mathematical community there is lack of uniformity regarding the concept of subnet. This is due to Willard and we will stick to this.

Example 1.1.16. Every subsequence is a subnet.

Example 1.1.17. Consider the sequence of natural numbers $\{x_n = n^2 + 1\}$. Then the net $\{y_{m,n}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ of natural numbers defined by $y_{m,n} = (m + n)^2 + 1$, is a subnet of the sequence $\{x_n\}$. To see this, note that we can take $\phi : \mathbb{N} \times \mathbb{N} \ni (m, n) \mapsto (m + n) \in \mathbb{N}$. Note that the net $\{y_{m,n}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ is not a subsequence of $\{x_n\}$.

Example 1.1.18. Net $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers. Let $N_0 = 0$. Let $\phi : \mathbb{N} \to \mathbb{N}$ be the map $\phi(j) = k$ if $n_{k-1} < j \leqslant n_k$. Then ϕ is monotone. Given any sequence $\{x_n\}$ consider the subnet $\{y_n\}_{n\in\mathbb{N}}$ given by $y_n = x_{\phi(n)}$. Clearly $\{y_n\}$ need not be a subsequence. For example if $n_k + 1 < n_{k+1}$, then $\{y_n\}$ is not a subsequence.

Example 1.1.19. Consider the nets $\{y_\lambda\}_{\lambda\in(0,1)}$ and $\{x_\alpha\}_{\alpha\in(0,\infty)}$ defined by $y_\lambda = \frac{1}{\lambda}$ where $\mu \preceq \lambda$ iff $\lambda \leqslant \mu$ and $x_\alpha = \alpha$ with $\alpha \preceq \beta$ iff $\alpha \leqslant \beta$. Then $\{y_\lambda\}$ and $\{x_\alpha\}$ are each other's subnet.

Definition 1.1.20 (Limit point/ Accumulation point/ Cluster point)**.** An element x in a topological space is a limit point (accumulation point/cluster pointare also used) of a net $\{x_\alpha\}$ if for all neighbourhood V of x and each index α there exists $\beta \succeq \alpha$ with $x_\beta \in V$. The possibly empty set of limit points of the net $\{x_\alpha\}$ is denoted by $Lim\{x_\alpha\}$.

Proposition 1.1.21. In a topological space X, a point x is a limit point of a net $\{x_\alpha\}$ iff x is the limit of some subnet of $\{x_\alpha\}$.

Proof. Let x be a limit point of the net $\{x_\alpha\}_{\alpha\in A}$. For each $(\alpha, V) \in A \times N_x$ pick some $\phi_{\alpha,V} \succeq \alpha$ and $x_{\phi_{\alpha,V}} \in V$. Now consider the net $\{y_{\alpha,V}\}$ given by $y_{\alpha,V} = x_{\phi_{\alpha,V}}$ and note that $\{y_{\alpha,V}\}$ is a subnet of $\{x_\alpha\}$ converging to x.

Conversely suppose a subnet $\{y_\beta\}_{\beta\in B}$ converges to x. Fix $\alpha_0 \in A$ and $V \in N_x$. We have to find $\alpha' \succeq \alpha_0$ such that $x_{\alpha'} \in V$. Since $\{y_\beta\}$ is a subnet we have a map $\phi : B \to A$. Since $\phi(B)$ is cofinal we can pick $\beta_0 \in B$ such that $\phi(\beta_0) \succeq \alpha_0$. Choose $\beta_1 \in B$ so that $\beta \succeq \beta_1 \implies y_\beta \in V$. Choose $\beta_2 \succeq \beta_1, \beta_2 \succeq \beta_0$. Then $\phi(\beta_2) \succeq \phi(\beta_0) \succeq \alpha_0$. Take $\alpha' = \phi(\beta_2)$. Then $x_{\alpha'} = x_{\phi(\beta_2)} = y_{\beta_2} \in V$. □

Remark 1.1.22. Is there a problem in the argument? Can you fix that? Don't read. Think and try.

Fixing the proof of proposition 1.1.21. Let x be a limit point of the net $\{x_\alpha\}_{\alpha\in A}$. Let $B = \{(\alpha, V) \in A \times N_x : x_\alpha \in V\}$ and $\phi : B \ni (\alpha, V) \mapsto \alpha \in A$. For each $(\alpha, V) \in A \times N_x$ pick some $\alpha' \succeq \alpha$ such that $x_{\alpha'} \in V$. Thus $(\alpha', V) \in B$ and $(\alpha', V) \succeq (\alpha, V)$. This shows $\phi(B)$ is a cofinal subset of the directed set A. Given $(\alpha, V) \in B$, let $\{y_{\alpha,V}\} = x_\alpha$. The net $\{y_\beta\}_{\beta\in B}$ is a subnet of $\{x_\alpha\}$. This subnet converges to x. □

Lemma 1.1.23. *In a topological space X, a net $\{x_\alpha\}$ converges to a point x iff every subnet converges to the same point.*

Proof. Obvious. □

Exercise 1.1.24. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in a topological space X. Suppose there exists $x \in X$ such that every subnet of the given net admits a further subnet converging to x. Then the original net $\{x_\lambda\}$ must be converging to x.

1.2 Filters

The concept of filters is due to Henry Cartan. This also serves the same purpose as nets in the sense that it allows us to describe the closure of a set in a topological space. One drawback of nets is the directed sets involved are not internal. This drawback is addressed in the notion of filters. Precise meaning of this remark will become clearer as we go along.

Definition 1.2.1. A filter on a set X is a family $\mathfrak{F}$ of subsets of X satisfying

1. $\emptyset \notin \mathfrak{F}$ and $X \in \mathfrak{F}$.
2. $A, B \in \mathfrak{F} \implies A \cap B \in \mathfrak{F}$.
3. If $A \subseteq B$ and $A \in \mathfrak{F}$ then $B \in \mathfrak{F}$.

A free filter is a filter $\mathfrak{F}$ with $\cap_{A\in\mathfrak{F}} = \emptyset$. The filters that are not free are called fixed.

Example 1.2.2. Let X be a set and $S \subseteq X$. Then $\mathfrak{F} := \{A \subseteq X | S \subseteq A\}$ is a filter. Note that this is a fixed filter.

Example 1.2.3. Let X be an infinite set and $\mathfrak{F} := \{A \subseteq X | A^c \text{ is finite }\}$ is a filter called the cofinite filter on X. This is a free filter.

Example 1.2.4 (Neighbourhood filter)**.** Let X be a topological space and $x \in X$. Then

$$\mathcal{N}_x := \{N | N \text{ is a neighbourhood of } x\}$$

is a filter called the neighbourhood filter at x.

Definition 1.2.5. A filter $\mathfrak{G}$ is a subfilter of another filter $\mathfrak{F}$ if $\mathfrak{F} \subseteq \mathfrak{G}$. Carefully note the nature of the inclusion for the term subfilter. In this case we say $\mathfrak{G}$ is finer than $\mathfrak{F}$. A filter $\mathfrak{U}$ is called an ultrafilter if it has no proper subfilter.

Our next result requires a set theoretic technology. We won't spend much time on this though.

Definition 1.2.6. A binary relation $\preceq$ on a set Λ is said to be antisymmetric if $\lambda_1 \preceq \lambda_2$ and $\lambda_2 \preceq \lambda_1$ for some $\lambda_1, \lambda_2 \in \Lambda$ implies $\lambda_1 = \lambda_2$. A reflexive, antisymmetric, transitive binary relation is called a partial order. A partially ordered set $(\Lambda, \preceq)$ is a pair consisting of a set Λ along with a partial order $\preceq$ on Λ. A subset $\Lambda' \subseteq \Lambda$ of a partially ordered set $(\Lambda, \preceq)$ is said to be linearly ordered if given any $\lambda, \lambda' \in \Lambda'$ we have either $\lambda \preceq \lambda'$ or $\lambda' \preceq \lambda$. In other words any two elements of Λ' can be compared. An element λ' is an upper bound for a subset Λ' of a partially ordered set $(\Lambda, \preceq)$ if $\lambda \preceq \lambda', \forall \lambda \in \Lambda'$.

Theorem 1.2.7. *[Zorn's lemma] Let $(\Lambda, \preceq)$ be a partially ordered set in which every linearly ordered subset has an upper bound. Then there is a $\lambda \in \Lambda$ which is maximal. This means that there is no $\lambda' \in \Lambda$ with $\lambda \preceq \lambda'$. Note that the theorem does not assert existence of an upper bound for Λ.*

Remark 1.2.8. Zorn's lemma is equivalent to the axiom of choice. In that sense it is an axiom of ZFC an axiomatic formulation of set theory.

Theorem 1.2.9. *Every filter is included in at least one ultrafilter. Consequently every infinite set has a free ultrafilter.*

Proof. Let $\mathfrak{F}$ be a filter on X and let $\mathcal{P}$ be the partially ordered set of all subfilters of $\mathfrak{F}$. $\mathcal{P}$ is partially ordered by inclusion. In this partially ordered set every linearly ordered subset $\mathcal{P}'$ has the obvious upper bound $\cup_{\mathfrak{F}\in\mathcal{P}'}\mathfrak{F}$. So Zorn's lemma applies and produces an ultrafilter as a maximal element of $\mathcal{P}$. For the last statement note that an ultrafilter containing the cofinite filter is free. □

Lemma 1.2.10. *Every fixed ultrafilter on a set* X *is of the form* $\mathfrak{U}_x = \{A \subseteq X | x \in A\}$.

Proof. Let $\mathfrak{U}$ be a fixed ultrafilter and let $x \in \cap_{A\in\mathfrak{U}} A$. Then $\mathfrak{U}_x$ is an ultrafilter containing $\mathfrak{U}$. Therefore $\mathfrak{U} = \mathfrak{U}_x$. □

Exercise 1.2.11. Let X be a set. Show that a collection $\mathfrak{F}$ of nonempty subsets of X closed under finite intersections is an ultrafilter iff for all subsets A of X, one of $A, X \setminus A$ belongs to $\mathfrak{F}$.

Definition 1.2.12. A nonempty collection $\mathfrak{B}$ of subsets of a set X is a filter base if

1. $\emptyset \notin \mathfrak{B}$.
2. If $A, B \in \mathfrak{B}$ then $\exists C \in \mathfrak{B}$ with $C \subseteq A \cap B$.

Every filter is a filter base. On the other hand, if $\mathfrak{B}$ is a filter base, then the collection of sets

$$\mathfrak{F}_{\mathfrak{B}} := \{A \subseteq X | B \subseteq A, \text{ for some } B \in \mathfrak{B}\}$$

is a filter, called the filter generated by $\mathfrak{B}$. For instance the open neighbourhoods of a point x of a topological space X forms a filter base $\mathfrak{B}$ generating $\mathcal{N}_x$.

Lemma 1.2.13. *An ultrafilter* $\mathfrak{U}$ *on a set* X *satisfies the following.*

1. *If* $A_1 \cup \cdots \cup A_n \in \mathfrak{U}$ *then* $A_i \in \mathfrak{U}$ *for some* i.
2. *If* $A \cap B \neq \emptyset, \forall B \in \mathfrak{U}$ *then* $A \in \mathfrak{U}$.

Proof. (1)Let $\mathfrak{U}$ be an ultrafilter on X and $A \cup B \in \mathfrak{U}$. If $A \notin \mathfrak{U}$ then $\mathfrak{F} := \{C | A \cup C \in \mathfrak{U}\}$ is a filter satisfying $B \in \mathfrak{F}$ and $\mathfrak{U} \subseteq \mathfrak{F}$. Hence $\mathfrak{U} = \mathfrak{F}$.

(2) Assume $A \cap B \neq \emptyset, \forall B \in \mathfrak{U}$. If we set $\mathfrak{B} := \{A \cap B | B \in \mathfrak{U}\}$, then $\mathfrak{B}$ is a filter base and $\mathfrak{U} \subseteq \mathfrak{F}_{\mathfrak{B}}$. So $\mathfrak{F}_{\mathfrak{B}} = \mathfrak{U}$. Since $A \in \mathfrak{F}_{\mathfrak{B}}$ we get $A \in \mathfrak{U}$. □

Lemma 1.2.14. *If $\mathfrak{U}$ is a free ultrafilter on a set X, then $\mathfrak{U}$ contains no finite subsets of X. In particular only infinite sets admit free ultrafilters.*

Proof. A free ultrafilter contains no singletons because if $\{x\} \in \mathfrak{U}$, then for any $A \in \mathfrak{U}, A \cap \{x\} \in \mathfrak{U}$. Since $\emptyset \notin \mathfrak{U}$, we must have $\{x\} \subseteq A$. Thus $x \in \cap_{A \in \mathfrak{U}} A$! Now for an ultrafilter if the finite set $\{x_1, \ldots, x_n\} = \cup_i \{x_i\} \in \mathfrak{U}$, then by the previous lemma we must have $\{x_i\} \in \mathfrak{U}$ for some i. ☐

Definition 1.2.15 (Filter Convergence)**.** A filter $\mathfrak{F}$ on a topological X converges to a point $x \in X$, written $\mathfrak{F} \to x$ is $\mathfrak{F}$ includes the neighbourhood filter $\mathcal{N}_x$ at x i.e., $\mathcal{N}_x \subseteq \mathfrak{F}$. Similarly a filter base $\mathfrak{B}$ converges to a point x, denoted $\mathfrak{B} \to x$ if $\mathfrak{F}_{\mathfrak{B}} \to x$. Clearly $\mathcal{N}_x \to x$.

Definition 1.2.16. An element x in a topological space X is a limit point of a filter $\mathfrak{F}$ if $x \in \overline{A}, \forall A \in \mathfrak{F}$. The set of all limit points is denoted by $\mathrm{Lim}\mathfrak{F}$. Clearly $\mathrm{Lim}\mathfrak{F} = \cap_{A \in \mathfrak{F}} \overline{A}$.

Proposition 1.2.17. In a topological space X a point x is a limit point of a filter iff there exists a subfilter converging to x.

Proof. Let $x \in \mathrm{Lim}\mathfrak{F} = \cap_{A \in \mathfrak{F}} \overline{A}$. Then $\mathfrak{B} = \{V \cap A | V \in \mathcal{N}_x, A \in \mathfrak{F}\}$. is a filter base. clearly $\forall A \in \mathfrak{F}, \forall V \in \mathcal{N}_x, V \cap A \subseteq A$, therefore $\mathfrak{F} \subseteq \mathfrak{F}_{\mathfrak{B}}$. Similarlr $\mathcal{N}_x \subseteq \mathfrak{F}_{\mathfrak{B}}$. That is $\mathfrak{F}_{\mathfrak{B}}$ is a subfilter converging to x.

For the converse, suppose $\mathfrak{F} \subseteq \mathfrak{G}$ is a subfilter with $\mathfrak{G} \to x$, i.e., $\mathcal{N}_x \subseteq \mathfrak{G}$. Then for each $A \in \mathfrak{F}$ and $V \in \mathcal{N}_x$ both belong to $\mathfrak{G}$ and consequently $V \cap A \neq \emptyset$. ☐

Exercise 1.2.18. In a topological space a filter converges to a point x iff every subfilter converges to x.

1.3 Relations between nets and filters

Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a net in a topological space X. For each $\lambda \in \Lambda$, let $F_\lambda := \{x_{\lambda'} : \lambda' \succeq \lambda\}$ and $\mathfrak{B} := \{F_\lambda : \lambda \in \Lambda\}$. Then $\mathfrak{B}$ is a filter base and $\mathfrak{F}_{\mathfrak{B}}$, the filter generated by $\mathfrak{B}$ is called the section filter of $\{x_\lambda\}$ or the filter generated by the net $\{x_\lambda\}$.

Proposition 1.3.1. Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a net in a topological space X. Then $\mathrm{Lim}\{x_\lambda\} = \mathrm{Lim}\mathfrak{F}$ where $\mathfrak{F}$ is the section filter of $\{x_\lambda\}$.

Proof. If $x \in \mathrm{Lim}\{x_\lambda\}$ then $y_\alpha \to x$ for some subnet $\{y_\alpha\}$. The filter generated by $\{y_\alpha\}$ is a subfilter of $\mathfrak{F}$ converging to x. Conversely if $x \in \mathrm{Lim}\mathfrak{F}$. Then consider the directed set

$$B = \{(\lambda, V) | \lambda \in \Lambda, V \in \mathcal{N}_x, x_\lambda \in V\}.$$

In other words B is a subset of the Cartesian product of Λ and $\mathcal{N}_x$ with the product direction. Define $\phi : B \to \Lambda$ by $\phi((\lambda, V)) = \lambda$, and $y_{(\lambda,V)} = x_\lambda$. Then $\{y_{(\lambda,V)}\}$ is a subnet of $\{x_\lambda\}$ converging to x. Therefore $x \in \mathrm{Lim}\{x_\lambda\}$. ☐

Consider an arbitrary filter $\mathfrak{F}$ in a topological space X. Define $\Lambda :=$ $\{(\lambda, A) | A \in \mathfrak{F}, \lambda \in A\}$. Then Λ has a natural direction given by $(\lambda, A) \succeq$ (λ', B) is $A \subseteq B$. We have a net $\{x_{(\lambda,A)} := \lambda\}$ with index set Λ. This net is called the net generated by the filter $\mathfrak{F}$. Observe that $F_{(\lambda,A)} = A$. So, the filter generated by $\{x_{(\lambda,A)}\}$ is $\mathfrak{F}$. In particular we have $\mathrm{Lim}\{x_{(\lambda,A)}\} = \mathrm{Lim}\mathfrak{F}$. We have proved the following theorem.

Theorem 1.3.2. *In a topological space X, a net and the filter it generates have the same limit points. Similarly a filter and the net it generates have the same limit points.*

1.4 Applications of nets and filters

Now we prove theorems reminiscent of results we had for metric spaces involving sequences.

Theorem 1.4.1. *For a function $f : X \to Y$ between topological spaces and points $x \in X$ the following are equivalent.*

1. *The function f is continuous at x.*
2. *If a net $x_\lambda \to x$ then $f(x_\lambda) \to f(x)$.*
3. *If a filter $\mathfrak{F} \to x$ then $f(\mathfrak{F}) \to f(x)$. Note that $f(\mathfrak{F})$ is a filter base.*

Proof. (1) implies (3): Let $\mathfrak{F} \to x$, i.e., $\mathcal{N}_x \subseteq \mathfrak{F}$. Continuity of f implies $f^{-1}(V) \in \mathcal{N}_x$ for each $V \in \mathcal{N}_{f(x)}$. From $f(f^{-1}(V)) \subseteq V$, we conclude that $\mathcal{N}_{f(x)}$ is included in the filter generated by $f(\mathfrak{F})$. Thus the filter base $f(\mathfrak{F})$ converges to $f(x)$.

(3) implies (2): Let $\{x_\alpha\}_{\alpha\in A}$ be a net converging to x. Let $\mathfrak{F}$ be the filter generated by the net $\{x_\alpha\}$. Then $\mathrm{Lim}\mathfrak{F} = \mathrm{Lim}\{x_\alpha\} = \{x\}$. In other words the filter $\mathfrak{F}$ converges to x. Therefore $f(\mathfrak{F})$ converges to $f(x)$. So $\mathcal{N}_{f(x)}$ is contained in the filter generated by $f(\mathfrak{F})$. Therefore if $V \in \mathcal{N}_{f(x)}$ there exists α_0 such that $f(\{x_\beta | \beta \succeq \alpha_0\}) \subseteq V$. Hence $\{f(x_\alpha)\}$ converges to $f(x)$.

(2) implies (1): Suppose f is not continuous at x. Then there is a neighbourhood V of $f(x)$ such that x is not in the interior of $f^{-1}(V)$. Then $x \in \overline{f^{-1}(V)^c}$. There exists a net $\{x_\alpha\} \subseteq f^{-1}(V)^c = f^{-1}(V^c)$ converging to x. Since $\{f(x_\alpha)\} \subseteq V^c$ and V^c is closed, $f(x) \in V^c$, a contradiction! □

Proposition 1.4.2. Let X be a compact topological space and $\{x_\alpha\}$ be a net in X. Then $\{x_\alpha\}$ has a convergent subnet. Equivalently we can show that every filter in X has a convergent subfilter. The converse is also true. That means if X is a topological space with the property that every filter on X has a convergent subfilter then X must be compact.

Proof. Let $\mathfrak{F}$ be a filter. Then $\mathfrak{G} := \{\overline{A} | A \in \mathfrak{F}\}$ has finite intersection property. So, $\mathrm{Lim}\mathfrak{F} = \cap_{A\in\mathfrak{F}}\overline{A} \neq \emptyset$. In other words $\mathfrak{F}$ has a convergent subfilter.

For the converse, let $\mathfrak{G}$ be a family of closed sets with the finite intersection property. Then finite intersections of elements of $\mathfrak{G}$ is a filter base. By hypothesis $\mathfrak{F}$, the filter generated by this filter base has a limit point. Therefore $\cap_{G\in\mathfrak{G}} G = \cap_{A\in\mathfrak{F}}\overline{A} = \mathrm{Lim}\mathfrak{F} \neq \emptyset$. □

1.5 Assignment-I, Due on 17/01/25

1. Let $\mathfrak{U}$ be an ultrafilter on X. Define $\mu_{\mathfrak{U}} : 2^X \to \mathbb{R}$, with $2 = \{0, 1\}$ by

$$\mu_{\mathfrak{U}}(A) = \begin{cases} 1 \text{ if } A \in \mathfrak{U} \\ 0 \text{ if } X \setminus A \in \mathfrak{U}. \end{cases}$$

Then $\mu_{\mathfrak{U}}$ is a finitely additive measure. Conversely if $\mu : 2^X \to \{0, 1\} \subseteq \mathbb{R}$ is a $\{0, 1\}$ valued finitely additive measure with $\mu(X) = 1$, then $\mathfrak{U}_\mu = \{A \subseteq X | \mu(A) = 1\}$ is an ultrafilter.

Definition 1.5.1. Let Λ be an index set and $\mathfrak{U}$ be an ultrafilter on Λ. Suppose we have a function $f : \Lambda \to Y$, where Y is a topological space. We say f has $\mathfrak{U}$-limit y for some y in Y if for all $V \in \mathcal{N}_y$, $f^{-1}(V) \in \mathfrak{U}$. This is denoted by $\mathfrak{U}\text{-}\lim f = y$.

2. Show that if Y is a compact Hausdorff space and $\mathfrak{U}$ is an ultrafilter on Λ then each $f : \Lambda \to Y$ has a $\mathfrak{U}$-limit. Determine $\mathfrak{U}$-limit for a principal ultra filter.

3. (Continued) This allows us to do funny things. For example if $\mathfrak{U}$ is an ultrafilter and $\{x_n\}$ is a bounded sequence in $\mathbb{R}$ then show that we can define $\mathfrak{U}\text{-}\lim x$. Or if we have a sequence $\{s_n\} \subseteq \mathbb{Z}_2$ where $\mathbb{Z}_2$ is the group with two elements with discrete topology. Then show that applying the previous exercise can define $\mathfrak{U}\text{-}\lim s$. Also show if $x \in \mathbb{Z}_2$, then the sequence $\{s'_n := x + s_n\}_n$ satisfies $\mathfrak{U}\text{-}\lim s' = x + \mathfrak{U}\text{-}\lim s$.

4. In a jail the jailer played the following game with the prisoners. All the prisoners were given T-shirts with a tick or a cross in the backside of the T-shirt and were arranged in a queue so that any prisoner could see the backsides of all the prisoners standing in front of him/her. Based on that he/she has to guess the mark on the T-shirt he/she is wearing. Let us assume prisoners are standing in positions 1,2, etc. so that the prisoner standing on position i can see the backsides of prisoners $i + 1, i + 2, \ldots$ etc. After the first prisoner declares his mark the second prisoner has to declare and so on. If everybody except the first prisoner can answer correctly then they will be released. Device a winning strategy.

5. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in A}$ be a family of topological spaces. Let $X = \prod_{\alpha\in A} X_\alpha$ be the product space with the product topology. Then a net $\{x_\lambda = (x_{\lambda,\alpha})_{\alpha\in A}\}_{\lambda\in\Lambda}$ in X converges to $x = (x_\alpha)_{\alpha\in A}$ iff $x_{\lambda,\alpha} \to x_\alpha, \forall \alpha \in A$.

6. (Stone-Cech Compactification) Let X be a discrete set. Then a filter is a subset of $\mathcal{P}(X) = 2^X$ or an element of 2^{2^X}. Therefore βX, the set of ultrafilters on X is a subset of 2^{2^X}. By Tychonoff's theorem 2^{2^X} is compact. Show that βX is closed and therefore compact.

7. (Continued) Consider the map $\mathfrak{U} : X \ni x \mapsto \mathfrak{U}_x \in \beta X$, where $\mathfrak{U}_x$ is the principal ultrafilter determined by x. Since $\mathfrak{U}$ is one to one we can and we will identify X as a subset of βX. Show that $\mathfrak{U}(X)$ is dense in βX.

8. (Continued) Finally, given any compact Hausdorff space Y and a map $f : X \to Y$ define $\tilde{f} : \beta X \to Y$ by $\tilde{f}(\mathfrak{U}) = \mathfrak{U}\text{-}\lim f$. Show that $\tilde{f}$ is continuous.

Solutions

(1.1.24) Let $\{x_\lambda\}_{\lambda\in\Lambda}\} \subseteq X$ be a net in a topological space X such that every subnet has a further subnet converging to $x \in X$. Now suppose $x_\lambda \nrightarrow x$. Consider the index set $B \coloneqq \{(\lambda, V) | \lambda \in \Lambda, V \in \mathcal{N}_x, x_\lambda \notin V\}$ with the binary relation $(\lambda, V) \preceq (\lambda', V')$ iff $\lambda \preceq \lambda', V \subseteq V'$. Equipped with this relation B is a directed set. Consider the net $\{y_{(\lambda,V)}\}_{(\lambda,V)\in B}$ given by $y_{(\lambda,V)} = x_\lambda$. Since $x_\lambda \nrightarrow x$,

$$\exists V_0 \in \mathcal{N}_x, \forall \lambda \in \Lambda, \exists \phi_\lambda \succeq \lambda, x_{\phi_\lambda} \notin V_0.$$

So, $(\phi_\lambda, V_0) \in B$ and if we define $\phi : B \ni (\lambda, V) \to \lambda \in \Lambda$, then ϕ is monotone and $\phi((\phi_\lambda, V_0)) = \phi_\lambda \succeq \lambda, \forall \lambda \in \Lambda$. In other words $\phi(B)$ is cofinal. Therefore $\{y_{(\lambda,V)}\}$ is a subnet of $\{x_\lambda\}$. Now we will show that the net $\{y_{(\lambda,V)}\}_{(\lambda,V)\in B}$ can't have a subnet converging to x. Or equivalently we will show that $x \notin \mathrm{Lim}\{y_{(\lambda,V)}\}_{(\lambda,V)\in B}$. That means we have to show that

$$\exists V' \in \mathcal{N}_x, \exists (\widetilde{\lambda}, \widetilde{V}) \in B, \forall (\lambda, V) \succeq (\widetilde{\lambda}, \widetilde{V}), y_{(\lambda,V)} \notin V'.$$

We will take $V' = V_0 = \widetilde{V}$ and $\widetilde{\lambda}$ such that $(\widetilde{\lambda}, \widetilde{V}) \in B$. We have already seen that there exists such $\widetilde{\lambda}$. Then for any $B \ni (\lambda, V) \succeq (\widetilde{\lambda}, \widetilde{V})$ we have $y_{(\lambda,V)} = x_\lambda \notin V \supseteq \widetilde{V} = V'$. Thus x is not a limit point of $\{y_{(\lambda,V)}\}_{(\lambda,V)\in B}$. But this contradicts the hypothesis that every subnet has a subnet converging to x.

(2) Let us assume on the contrary that f has no ultra limit. Then $\forall y \in Y, \exists V_y \in \mathcal{N}_y$ with $f^{-1}(V_y) \notin \mathfrak{U}$. Since $\mathfrak{U}$ is an ultrafilter given any set it must contain that set or it's complement. Therefore we conclude that $\Lambda \setminus f^{-1}(V_y) \in \mathfrak{U}$. Appealing to the compactness of Y we obtain $y_1, \ldots, y_n$ so that $Y = \cup_{i=1}^n V_{y_i}$. Then $\Lambda = \cup_{i=1}^n f^{-1}(V_{y_i})$. Therefore $\emptyset = \cap_{i=1}^n (\Lambda \setminus f^{-1}(V_{y_i}) \in \mathfrak{U}$, a contradiction!

(4) Let us define $x_n = 1$ if the i-th prisoner gets a T-shirt marked with a tick, else $x_n = 0$. The variables x_n takes values in the group $\mathbb{Z}_2$. Fix a free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$. For each $k \in \mathbb{N}$ consider the sequence s_k whose n-th term is $s_{k,n} \coloneqq \sum_{j=k+1}^{k+n} x_j \subseteq \mathbb{Z}_2$. If we denote by y_k the response of the k-th prisoner. Let us consider the strategy where

the first player announces $\mathfrak{U}\text{-}\lim s_1$ and for $k > 1, y_k = \sum_{j=1}^{k-1} y_j + \mathfrak{U}\text{-}\lim s_k$.

(6) Let $\{\mathfrak{U}_\lambda\}$ be a net in βX converging to $\mathfrak{U}$. We have to show that $\mathfrak{U} \in \beta X$. We first show that $\mathfrak{U}$ is a filter. Note that $\mu_{\mathfrak{U}_\lambda}(B) \to \mu_{\mathfrak{U}}(B), \forall B \subseteq X$. Therefore empty set belongs to $\mathfrak{U}$ because $\mu_{\mathfrak{U}}(\emptyset) = \lim_\lambda \mu_{\mathfrak{U}_\lambda}(\emptyset) = 0$. Similarly we can verify other filter properties. Finally to show that it is an ultrafilter let $A \subseteq X$. We have to show that $\mu_{\mathfrak{U}}(A) + \mu_{\mathfrak{U}}(A^c) = 1$. That follows from $\mu_{\mathfrak{U}}(A) + \mu_{\mathfrak{U}}(A^c) = \lim_\lambda(\mu_{\mathfrak{U}_\lambda}(A) + \mu_{\mathfrak{U}_\lambda}(A^c)) = 1$.

(7) Let $\mathfrak{U}$ be an ultrfilter. We will exhibit a net from $X \subseteq \beta X$ converging to $\mathfrak{U}$. Let $\Lambda := \{(a, A) | A \in \mathfrak{U}, a \in A\}$. Consider the direction on Λ defined by $(a, A) \succeq (b, B)$ if $A \subseteq B$. Consider the net $\{\mathfrak{U}_{(a,A) \in \Lambda}$ given by $\mathfrak{U}_{(a,A)} = \mathfrak{U}_a$, the principal ultrfilter. Now we will show that $\mathfrak{U}_{(a,A)} \to \mathfrak{U}$ or equivalently $\mu_{\mathfrak{U}_{(a,A)}}(B) \to \mu_{\mathfrak{U}}(B), \forall B \subseteq X$.

Let us fix $B \subseteq X$. Suppose $B \in \mathfrak{U}$. Take $\lambda_0 = (b, B) \in \Lambda$. Then for $(b', B') \succeq (b, B)$ we have $B' \subseteq B$. Therefore $b' \in B' \subseteq B$. Consequently $B \in \mathfrak{U}_{b'} = \mathfrak{U}_{(b',B')}$. Therefore $\mu_{\mathfrak{U}_{(b',B')}} = 1$. If $B \notin \mathfrak{U}$ we use $B^c \in \mathfrak{U}$.

(8) Let $\{\mathfrak{U}_\lambda\}$ be a net in βX converging to $\mathfrak{U}$. If we denote $\mathfrak{U}_\lambda\text{-}\lim f$ by y_λ and $\mathfrak{U}\text{-}\lim f$ by y, then we have to show that $\lim_\lambda y_\lambda = y$. Suppose that the net $\{y_\lambda\}$ does not converge to y. That means there is a neighbourhood V of y and a subnet of $\{y_\lambda\}$ that lies outside V (why?). If necessary by passing to the subnet we can assume that the net $\{y_\lambda\}$ lies outside V. If necessary by passing to a further subnet we can assume $y_\lambda \to y'$. Get an open neighbourhood V' of y' disjoint from V. Since $\mathfrak{U}\text{-}\lim f = y$ we have $f^{-1}(V) \in \mathfrak{U}$. Therefore from $1 = \mu_{\mathfrak{U}}(f^{-1}(V)) = \lim_\lambda \mu_{\mathfrak{U}_\lambda}(f^{-1}(V))$ we conclude that there exists λ_0 such that $\forall \lambda \succeq \lambda_0, \mu_{\mathfrak{U}_\lambda}(f^{-1}(V)) = 1$ or $f^{-1}(V) \in \mathfrak{U}_\lambda$. On the other hand since $y_\lambda \in V'$ for a cofinal set of indices we conclude that eventually $f^{-1}(V') \in \mathfrak{U}_\lambda$. Therefore $\emptyset = f^{-1}(V') \cap f^{-1}(V) \in \mathfrak{U}_\lambda$ for a cofinal set of λ's. This contradiction establishes that the assertion $\{y_\lambda\}$ does not converge to y must be wrong.

Chapter 2

A very general setup

We introduce the most general framework for doing functional analysis. These are vector spaces endowed with topologies that makes addition and scalar multiplication continuous. We extract properties of such topologies. This in turn allows us to characterize such topologies. Then we see characterization of finite dimensional vector spaces. See some more examples and learn about limitations of making things too general.

2.1 Topological Vector Spaces

In this course we will use $\mathbb{K}$ to mean a statement which holds for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.1 (Linear Topology/ Vector Space Topology)**.** Let E be a vector space over $\mathbb{K}$. A topology $\mathcal{T}$ on E is called a linear topology or vector space topology if

1. the operation of addition $+ : E \times E \ni (a, b) \mapsto (a + b) \in E$ and

2. the operation of scalar multiplication $. : \mathbb{K} \times E \ni (\alpha, x) \mapsto \alpha.x \in E$

are continuous. Here we endow $E \times E$ and $\mathbb{K} \times E$ with the product topologies.

Definition 2.1.2 (Topological Vector Space)**.** A vector space equipped with a linear topology is called a topological vector space, TVS in short.

Exercise 2.1.3. Show that $(E, \mathcal{T})$ is a TVS iff

(i) for every pair of convergent nets $\{x_\lambda\}, \{y_\lambda\} \subseteq E$ with $x_\lambda \to x, y_\lambda \to y$ we have $x_\lambda + y_\lambda \to (x+y)$;

(ii) for every pair of convergent nets $\{a_\lambda\} \subseteq \mathbb{K}, \{x_\lambda\} \subseteq E$ with $a_\lambda \to a, x_\lambda \to x$ we have $a_\lambda . x_\lambda \to a.x$.

Remark 2.1.4. Let E be a TVS

(i) for all $y \in E$, the translation $T_y : E \ni x \mapsto (x+y) \in E$;

(ii) for all $\alpha \in \mathbb{K} \setminus \{0\}$, the dilations $D_\alpha : E \ni x \mapsto \alpha.x \in E$

are homeomorphisms.

We wish to understand vector space topologies or which topologies on a $\mathbb{K}$-vector space will turn that into a TVS. For that purpose we begin by exploring some of the necessary conditions of the neighbourhood base at origin.

Proposition 2.1.5. Let E be a TVS.

1. for any neighbourhood V of $0 \in E$, there exists a neighbourhood W of $0 \in E$ such that $\{x+y : x, y \in W\} =: W + W \subseteq V$;

2. for any neighbourhood V of $0 \in E$ and any compact set $C \subseteq \mathbb{K}$ there exists a neighbourhood W of $0 \in E$ such that $\{\alpha.x | \alpha \in C, x \in W\} =: C.W \subseteq V$.

Proof. (i) Since addition is continuous and $0+0$ is 0, there exists $W_1, W_2 \in \mathcal{N}_0$ such that $\{x+y | x \in W_1, y \in W_2\} =: W_1 + W_2 \subseteq V$. Take $W_1 \cap W_2$ as W.

(ii) Similarly using the continuity of scalar multiplication we get a neighbourhood W_1 of origin in $\mathbb{K}$ and a neighbourhood W_2 of origin in E such that $W_1.W_2 \subseteq V$. The result now follows from locally compactness of $\mathbb{K}$ and the fact that dilations are homeomorphisms. □

Definition 2.1.6. A subset $A \subseteq E$ is said to be absorbing if $\forall x \in E, \exists \lambda > 0$ such that $\lambda.x \in A$.

A subset $A \subseteq E$ is said to be balanced if $x \in A, \alpha \in \mathbb{K}$ with $|\alpha| \leqslant 1$ implies $\alpha.x \in A$. Balanced hull of a subset A is the smallest balanced set containing A. This is given by $\cup_{\alpha:|\alpha|\leqslant 1} \alpha.A$ and is denoted by Bal A.

Remark 2.1.7. Any neighbourhood of origin in a TVS E is absorbing because given any $x \in E, \{\frac{x}{n}\}_n$ is converging to 0.

The next proposition gives a reasonable list of necessary conditions for a linear topology.

Proposition 2.1.8. Let E be a TVS.

(i) If $\mathfrak{B}$ is a basic system of neighbourhoods at 0, then

  (a) for all $V \in \mathfrak{B}$, there exists $W \in \mathfrak{B}$ such that $W + W \subseteq V$;

  (b) for all $V \in \mathfrak{B}$, for all compact $C \subseteq \mathbb{K}$, there exists $W \in \mathfrak{B}$ such that $C.W \subseteq V$.

  (c) for all $x \in E$, $\mathfrak{B}_x := \{x + V | V \in \mathfrak{B}\}$ is a local base at x.

  (d) The topology of E is Hausdorff iff $\cap_{V \in \mathfrak{B}} V = \{0\}$.

(ii) There exists a basic system of neighbourhoods of 0 consisting of open balanced sets.

Proof. (i) We have already seen proofs of (a), (b). Proof of (c) follows from the fact that translations are homeomorphisms. Let us prove (d). Let $W = \cap_{V \in \mathfrak{B}} V$. Then $0 \in W$. Assume that the topology is Hausdorff. Then for all $x \in E \setminus \{0\}$, the set $E \setminus \{x\}$ is an open neighbourhood of origin. So, there exists $V_x \in \mathfrak{B}$ such that $V_x \subseteq E \setminus \{x\}$. Then

$$W \subseteq \cap_{x \neq 0} V_x \subseteq \cap_{x \neq 0}(E \setminus \{x\}) = \{0\}.$$

Conversely suppose $W = \{0\}$. Let $x \neq y$ be two elements of E. without loss of generality we can assume $y = 0$. Since x is nonzero, $x \notin \cap_{V \in \mathfrak{B}} V$. Therefore there exists $V \in \mathfrak{B}$ such that $x \notin V$. By (a) there exists $W \in \mathfrak{B}$ such that $W + W \subseteq V$. So, $x \notin W + W$. This implies $(x + (-1).W) \cap W = \emptyset$.
(ii) Let $\mathfrak{B}$ be the collection of all balanced sets containing 0. We know that for any neighbourhood V of 0 there exists an open W such that $\gamma.W \subseteq V$ for all γ with $|\gamma| \leqslant 1$. Then Bal W is an open, balanced subset of V. □

Definition 2.1.9. Let E be a TVS. A subset $B \subseteq E$ is said to be bounded if for all neighbourhood V of $0 \in E$, there exists $\rho > 0$ such that $B \subseteq \rho V$.

Proposition 2.1.10. Let E be a TVS. If the net $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq \mathbb{K}$ converges to $0 \in \mathbb{K}$ and the net $\{x_\lambda\}_{\lambda \in \Lambda} \subseteq E$ is bounded, then $\{a_\lambda x_\lambda\}$ converges to $0 \in E$.

Proof. Let V be a neighbourhood of origin in E. Without loss of generality we can assume V is balanced. Using the boundedness condition we get $x_\lambda \in \rho V, \forall \lambda \in \Lambda$. Since $\{a_\lambda\}$ is converging to zero we get $|a_\lambda| < \frac{1}{\rho}, \forall \lambda \succeq \lambda_0$. Since V is balanced $a_\lambda x_\lambda \in V, \forall \lambda \succeq \lambda_0$. □

Exercise 2.1.11. Show that the last proposition holds if we assume $\{x_\lambda\}$ is eventually bounded, i.e., there exists λ_0 so that $\{x_\lambda | \lambda \succeq \lambda_0\}$ is bounded.

Theorem 2.1.12. *Suppose E is a $\mathbb{K}$-vector space and $\mathfrak{B}$ is a filter base on E with the following properties:*

(i) $0 \in V, \forall V \in \mathfrak{B}$;

(ii) every $V \in \mathfrak{B}$ *is absorbing;*

(iii) for every $V \in \mathfrak{B}, \exists W \in \mathfrak{B}$ *satisfying* $W + W \subseteq V$;

(iv) for every $V \in \mathfrak{B}, \exists W \in \mathfrak{B}, r > 0$ *so that* $\rho.W \subseteq V, \forall \rho \in \mathbb{K}$ *with* $|\rho| \leqslant r$.

Then there exists a unique linear topology on E so that $\mathfrak{F}_{\mathfrak{B}}$ is the neighbourhood filter at origin of E. Moreover the topology is Hausdorff iff $\{0\} = \cap_{V \in \mathfrak{B}} V$.

Proof. Declare a set $A \subseteq E$ to be open if

$$\forall a \in A, \exists V \in \mathfrak{B}, V + a \subseteq A.$$

Let $\mathcal{T}$ be the collection of all open sets. We will begin by showing that $\mathcal{T}$ is a topology. The collection $\mathcal{T}$ contains $\emptyset$ vacuously and $E \in \mathcal{T}$. Also it is obvious that $\mathcal{T}$ is closed under arbitrary unions. Let us check that $\mathcal{T}$ is closed under finite intersections. Let $A, B \in \mathcal{T}$ and $a \in A \cap B$. There exists $V_1, V_2 \in \mathfrak{B}$ such that $a + V_1 \subseteq A$, $a + V_2 \subseteq B$. Since $\mathfrak{B}$ is a filter base we can get $V \subseteq V_1 \cap V_2, V \in \mathfrak{B}$. Then $a + V \subseteq a + V_1 \subseteq A$ and $a + V \subseteq a + V_2 \subseteq B$. Therefore $a + V \subseteq A \cap B$. This shows $A \cap B \in \mathcal{T}$. This completes proof of the fact that $\mathcal{T}$ is a topology.

Next we will show that the filter $\mathfrak{F}_{\mathfrak{B}}$ generated by the filter base $\mathfrak{B}$ is the neighbourhood filter at origin of the topology $\mathcal{T}$. This is done in the following steps.

- Given any $B \in \mathfrak{B}$, define $A = \{x \in E | \exists V \in \mathfrak{B}, V + x \subseteq B\}$.

- Since $0 \in B$, we can take $V = B$ to conclude that $0 + V \subseteq B$. Thus $0 \in A$.

- To see that A is open, given $a \in A$ we need to exhibit $V'' \in \mathfrak{B}$ so that $a + V'' \subseteq B$. Since $a \in A$, there exists $V' \in \mathfrak{B}$ so that $a + V' \subseteq B$. On the other hand by (iii) there exists $V'' \in \mathfrak{B}, V'' + V'' \subseteq V'$. Then $a + V'' + V'' \subseteq B$. In particular $a + V'' \subseteq A$.

- The A constructed above is contained in B. To see this note that if we take a from A then $a \in B$ because every element of $\mathfrak{B}$ contains origin.

- So we have proved $B \supseteq A$ and A is an open set containing origin. So, B is a neighbourhood of origin. Thus $\mathfrak{B}$ is a subset of the neighbourhood filter at origin. Therefore $\mathfrak{F}_{\mathfrak{B}} \subseteq \mathcal{N}_0$.

- The other inclusion $\mathcal{N}_0 \subseteq \mathfrak{F}_{\mathfrak{B}}$ is immediate because if B is a neighbourhood of origin then there exists an open set A containing origin and $A \subseteq B$. Since A is open and $0 \in A$ there exists $V \in \mathfrak{B}$ with $V \subseteq A$. Being a superset of an element of $\mathfrak{B}$, we have $B \in \mathfrak{F}_{\mathfrak{B}}$.

Now we will show continuity of addition. Suppose we have nets $\{x_\lambda\}_\Lambda, \{y_\lambda\}_\Lambda$ converging to x, y respectively. We have to show given any open neighbourhood A' of $(x + y)$ there exists λ_0 so that $x_\lambda + y_\lambda \in A', \forall \lambda \succeq \lambda_0$.

- Clearly for every $u \in E$, a subset $A \subseteq E$ is open iff $A + u$ is open. Therefore A', an open neighbourhoods of $x + y$ is of the form $A + (x + y)$ for some open neighbourhood A of origin. So, we have to show that given any open neighbourhood of origin A, there exists λ_0 such that $x_\lambda + y_\lambda \in A + (x + y), \forall \lambda \succeq \lambda_0$.

- Since open neighbourhoods of origin are supersets of elements of $\mathfrak{B}$, it suffices to show $\forall V \in \mathfrak{B}$, there exists λ_0 with $x_\lambda + y_\lambda \in V + (x + y), \forall \lambda \succeq \lambda_0$. By property (iii) of our hypothesis there exists $W \in \mathfrak{B}$ satisfying $W + W \subseteq V$. Obtain λ_0 such that $\lambda \succeq \lambda_0$ implies $x_\lambda \in W + x, y_\lambda \in W + y$. Then for $\lambda \succeq \lambda_0$ we have $x_\lambda + y_\lambda \in W + W + (x + y) \subseteq V + (x + y)$. This completes proof of continuity of addition.

To show that $\mathcal{T}$ is a vector space topology only thing remains to be shown is the continuity of scalar multiplication. Let $\{x_\lambda\} \subseteq E, \{\alpha_\lambda\} \subseteq \mathbb{K}$ be nets with $x_\lambda \to x, \alpha_\lambda \to \alpha$. We have to show that $\alpha_\lambda . x_\lambda \to \alpha . x$. Using continuity of addition it is enough to show that

(a) $\alpha_\lambda(x_\lambda - x) \to 0$;

(b) $(\alpha_\lambda - \alpha).x \to 0$.

Let $V \in \mathfrak{B}$. Fix $r > 0, W \in \mathfrak{B}$ be as in (iv) of our hypothesis involving $\mathfrak{B}$.

Proof of (a). Since $\alpha_\lambda \to \alpha$, there exists λ_0 so that $M := \sup_{\lambda \succeq \lambda_0} |\alpha_\lambda| < \infty$. Let $n \in \mathbb{N}$ be such that $nr > M$. Let $Z \in \mathfrak{B}$ be such that $n.Z \subseteq Z + \cdots + Z \subseteq W$. For all $\lambda \succeq \lambda_0$, we have $|\alpha_\lambda / n| \leqslant r$. Therefore

$$\alpha_\lambda . Z = \frac{\alpha_\lambda}{n} . nZ \subseteq \frac{\alpha_\lambda}{n} . W \subseteq V, \forall \lambda \succeq \lambda_0.$$

Since $x_\lambda \to x$, there exists λ_1 such that $x_\lambda - x \in Z, \forall \lambda \succeq \lambda_1$. If we take $\lambda_V \succeq \lambda_1, \lambda_0$, then for all $\lambda \succeq \lambda_V$ we have $\alpha_\lambda(x_\lambda - x) \in V$. □

Proof of (b). Since every $V \in \mathfrak{B}$ is absorbing we can choose $t > 0$ so that $t.x \in W$. We choose ν_V so that $|\alpha_\lambda - \alpha| \leqslant rt, \forall \lambda \succeq \nu_V$. Equivalently $|t^{-1}(\alpha_\lambda - \alpha)| \leqslant r, \forall \lambda \succeq \nu_V$. So, by condition (iv), $t^{-1}(\alpha_\lambda - \alpha).W \subseteq V, \forall \lambda \succeq \nu_V$. Since $t.x \in W$, $(\alpha_\lambda - \alpha).x = t^{-1}(\alpha_\lambda - \alpha).tx \in V, \forall \lambda \succeq \nu_V$. □

Uniqueness is obvious and we have seen the characterization of the Hausdorff property before. □

Corollary 2.1.13. Suppose E is a $\mathbb{K}$-vector space and $\mathfrak{B}$ is a filter base on E with the following properties:

(i) $0 \in V, \forall V \in \mathfrak{B}$;

(ii) every $V \in \mathfrak{B}$ is absorbing;

(iii) for every $V \in \mathfrak{B}, \exists W \in \mathfrak{B}$ satisfying $W + W \subseteq V$;

(iv) every $V \in \mathfrak{B}$ is balanced.

Then there exists a unique linear topology on E so that $\mathfrak{F}_{\mathfrak{B}}$ is the neighbourhood filter at origin of E. Moreover the topology is Hausdorff iff $\{0\} = \cap_{V \in \mathfrak{B}} V$.

Proof. The theorem applies because condition (iv) of the theorem holds with $r = 1$ and $W = V$. □

Practice Problems

Let E be a topological vector space.

1. Prove that if $A \subseteq E$ is open then so is $A + B$ for any subset B.

2. Prove that if V is a neighbourhood of origin then for any subset A we have $\overline{A} \subseteq A + V$.

3. Show that given any neighbourhood V of origin there exists a balanced, closed neighbourhood W of origin, such that $W \subseteq V$.

4. Let $F = \overline{\{0\}}$ be the closure of the set $\{0\}$. Prove that

   (i) F equals the intersection of all open neighbourhoods of origin.

   (ii) F is a closed linear subspace.

   (iii) F is compact.

5. Show that if $A \subseteq E$ is closed and $C \subseteq E$ is compact, then $A + C$ is closed. Is this result true if we take C to be closed?

6. Prove that if $A, C \subseteq E$ are compact then so is $\cup_{\gamma \in C} \gamma.A$.

7. Show that if $A \subseteq E$ is closed and $C \subseteq \mathbb{K} \setminus \{0\}$ is compact, then $C.A := \{\gamma.a | \gamma \in C, a \in A\}$ is closed. Give a counter example to show that the condition $0 \notin C$ is essential.

8. Show that if $A, B \subseteq E$ are compact then so is $A + B$.

9. Let F be a TVS and $T : E \to F$ is linear. Prove that the following are equivalent.

   (a) T is continuous.

   (b) T is continuous at 0.

2.2 Finite dimensional topological vector spaces

We have introduced a concept but so far haven't discussed any examples. Of course finite dimensional vector spaces with their usual topologies are topological vector spaces and Tychonoff showed the converse.

Theorem 2.2.1 (Tychonoff)**.** *Every finite dimensional Hausdorff topological vector space has the usual topology.*

Proof. Let E be a finite dimensional Hausdorff TVS with the topology $\mathcal{T}$. Let $v_1, \ldots, v_n$ be a basis of E. Let $T : \mathbb{R}^n \to E$ be the map $T(x_1, \ldots, x_n) = \sum x_i v_i$. This is a bijection and using this we can identify E with $\mathbb{R}^n$. This means we consider the topology $T^{-1}(\mathcal{T})$ on $\mathbb{R}^n$. So we have two topologies on $\mathbb{R}^n$, the usual one and another vector space topology, to be denoted $\mathcal{T}$, not $T^{-1}(\mathcal{T})$. We will use $T : (\mathbb{R}^n, \text{product topology}) \to (\mathbb{R}^n, \mathcal{T})$ to denote the identity map, considered as a map with the indicated topologies. Since $\mathcal{T}$ is a linear topology, the operations of scalar multiplication and addition are continuous. Therefore so is T because $T(\mathbf{x}) = \sum x_i e_i$ where e_i's are the canonical basis elements. This shows any open set in $\mathcal{T}$ is open in the usual product topology. As T is continuous any compact set in the usual topology is compact in $\mathcal{T}$. In particular $\mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n | \|\mathbf{x}\|_2 = 1\}$ is compact in $\mathcal{T}$. Since $\mathcal{T}$ is Hausdorff and $0 \notin \mathbb{S}^{n-1}$, we can find an open set U in $\mathcal{T}$ containing origin and disjoint from $\mathbb{S}^{n-1}$. We wish to show U is inside the unit ball. Or it will be enough to exhibit an open set in $\mathcal{T}$ containing the origin and inside a ball. That is achieved as follows. Using the continuity of the scalar multiplication map we can find another open $U' \in \mathcal{T}$ containing origin and an $\epsilon > 0$ such that $(-\epsilon, \epsilon)U' \subseteq U$. Since U is disjoint from $\mathbb{S}^{n-1}$, we have for all $t \in (-\epsilon, \epsilon)$ and for all $\mathbf{x} \in U'$, $\|t.\mathbf{x}\|_2 \neq 1$. In other words $\forall \mathbf{x} \in U', \|\mathbf{x}\|_2 \notin \{\frac{1}{|t|} | 0 < |t| < \epsilon\}$. Therefore for all $\mathbf{x} \in U', \|\mathbf{x}\|_2 \leqslant \frac{1}{\epsilon} < \frac{2}{\epsilon}$. So, $\mathcal{T}$ contains a neighbourhood U' contained in the open ball of radius $\frac{2}{\epsilon}$. This shows T^{-1} is continuous at origin. Since linear topologies are translation invariant and T is linear we conclude that T is continuous at any other point. □

Corollary 2.2.2. In a Hausdorff TVS E every finite dimensional subspace F is closed.

Proof. Let $\{x_\lambda\}$ be a net in F converging to x. Let us consider the span of F and x. This is finite dimensional and therefore has the usual topology. But a subspace of a finite dimensional space is closed. Therefore x must be in F. □

This shows if we are looking for more examples we must consider infinite dimensional spaces. This course is primarily about them Before we venture into that world we ask ourselves is there a topological characterization of finite dimensionality. This is addressed in the following theorem

Theorem 2.2.3 (André Weil)**.** *Every locally compact, Hausdorff TVS is finite dimensional.*

Proof. Let E be a locally compact, Hausdorff TVS. Thus there exists a compact neighbourhood C of origin. Then the dilate $\frac{1}{2}C$ is also a compact neighbourhood of origin. Using compactness of C we conclude that finitely many translates of $\frac{1}{2}C$ covers C. Thus there exists a finite set S so that $C \subseteq S + \frac{1}{2}C$. Let F be the linear span of S. We have just proved that being finite dimensional F is closed. It suffices to show that C is a subset of F. That will follow once we show that given any open neighbourhood of origin V, $C \subseteq F + V$. Note that $C \subseteq F + \frac{1}{2}C$. Iterating this we get $C \subseteq F + \frac{1}{2^n}C$ for all $n \in \mathbb{N}$. Using continuity of scalar multiplication we conclude that for all $x \in E$, there exists $\epsilon_x > 0$ and V_x an open neighbourhood of x satisfying $t.V_x \subseteq V$ for all t with $|t| < \epsilon_x$. There exists $x_1, \ldots, x_n$ so that $C \subseteq \cup_{i=1}^n V_{x_i}$. Choose N large enough with $2^{-N} < \epsilon_{x_i}, \forall i$. Let $x \in C$ be arbitrary. Then $x \in V_{x_i}$ for some i. We have $2^{-N}x \in V$. Thus $2^{-N}C \subseteq V$. Therefore $C \subseteq F + V$. Since V is an arbitrary neighbourhood we conclude that $C \subseteq \overline{F}$. But F being finite dimensional is closed. □

2.3 Examples of TVS

It is customary that any definition is accompanied by a preferably long list of examples. We will also do so, but little later. We won't spend much time in this generality, instead soon we will concentrate on a convenient subclass of topological vector spaces called locally convex spaces. To motivate introduction of such a subclass we begin with examples some of which aren't particularly nice.

Example 2.3.1. Fix $p \in (0, \infty)$. Define

$$\ell^p_{\mathbb{K}} = \{x = (x_n)_{n\in\mathbb{N}} : \sum |x_n|^p < \infty\}$$

The inequality $(a+b)^p \leqslant 2^p(a^p + b^p), \forall a, b \geqslant 0$ implies $\ell^p_{\mathbb{K}}$ is a $\mathbb{K}$ vector space. For all $r > 0$ define $V_r \coloneqq \{x | \sum |x_n|^p < r\}$. The collection $\mathfrak{B} = \{V_r | r > 0\}$ is a filter base on $\ell^p_{\mathbb{K}}$ each of whose elements contain origin, absorbing and balanced. Let us verify condition (iii) of the hypothesis of corollary 2.1.13. Let $(x_n), (y_n) \in V_{r/2^{p+1}}$. Then

$$\begin{aligned} \sum |x_n + y_n|^p &\leqslant \sum (|x_n| + |y_n|)^p \\ &\leqslant 2^p \sum (|x_n|^p + |y_n|^p) \\ &< 2^{p+1} \frac{r}{2^{p+1}} = r. \end{aligned}$$

Therefore $V_{r/2^{p+1}} + V_{r/2^{p+1}} \subseteq V_r$. So by corollary 2.1.13 $\mathfrak{B}$ generates a unique vector space topology on $\ell^p_{\mathbb{K}}$.

Example 2.3.2. Let $(\Omega, \mathfrak{S}, \mu)$ be a measure space. Let $0 < p$. Define

$$\mathcal{L}^p(\Omega, \mathfrak{S}, \mu) \coloneqq \{f : \Omega \to \mathbb{K} | f \text{ is measurable and } \int |f|^p d\mu < \infty\}.$$

For all $r > 0$ define $V_r \coloneqq \{f \in \mathcal{L}^p(\Omega, \mathfrak{S}, \mu) | \int |f|^p d\mu < r\}$. The collection $\mathfrak{B} = \{V_r | r > 0\}$ is a filter base on $\mathcal{L}^p(\Omega, \mathfrak{S}, \mu)$. As in the last example replacing sum by integrals we conclude that $\mathfrak{B}$ generates a vector space topology.

Answering what is bad about these examples takes us to the important concept of the dual of a TVS.

2.4 Dual of a TVS

Recall that in point set topology first nontrivial result one learns is the Urysohn lemma guaranteeing the existence of nontrivial continuous functions. We are also after something similar. Let us begin with a closer look at continuous linear maps.

Theorem 2.4.1. *Let E be a TVS over $\mathbb{K}$ and $\phi : E \to \mathbb{K}$ be a nonzero linear map. Then the following are equivalent.*

(i) ϕ *is continuous;*

(ii) $\ker(\phi)$ *is closed;*

(iii) $\ker(\phi)$ *is not dense in* E*;*

(iv) there exists a neighbourhood V *of* $0 \in E$ *such that* $\phi(V)$ *is bounded or equivalently* ϕ *is bounded on* V*;*

(v) there exists an open neighbourhood V *of* $0 \in E$ *so that* $\phi(V) \neq \mathbb{K}$*.*

(vi) If $\mathbb{K} = \mathbb{C}$ *then these are equivalent to* $\mathfrak{R}\phi$ *is continuous.*

Proof. $(i) \implies (ii), (iv) \implies (v)$ are trivial.

$(ii) \implies (iii)$: If $E = \overline{\ker(\phi)} = \ker(\phi)$, the last equality holds because $\ker(\phi)$ is closed. Then $\phi \equiv 0$. This contradicts that ϕ is nonzero.

$(iii) \implies (iv)$: Choose $x \in E$ and a balanced neighbourhood U of 0 such that $(x + U) \cap \ker(\phi) = \emptyset$. Therefore $\phi(x) \notin -\phi(U)$. But $\phi(U)$ is balanced as U is balanced and hence $\phi(U)$ is bounded because a proper balanced subset of $\mathbb{K}$ is bounded.

$(v) \implies (i)$: Assume that V is a balanced neighbourhood of origin and $\phi(V) \neq \mathbb{K}$. Since $\phi(V)$ is also balanced it must be bounded. So, there exists $M > 0$ so that $\phi(V) \subseteq B_{\mathbb{K}}(0, M)$, where $B_{\mathbb{K}}(0, M) = \{\alpha \in \mathbb{K} | |\alpha| < M\}$. Then for any $\epsilon > 0, \phi(\frac{\epsilon}{M} V) \subseteq B_{\mathbb{K}}(0, \epsilon)$ establishing continuity of ϕ at origin. Since ϕ is linear, this shows ϕ is continuous.

$(vi) \iff (i)$: This follows from the simple fact that if $(\mathfrak{R}\phi)(x), (\mathfrak{I}\phi)(x) \in \mathbb{R}$ are defined by $\phi(x) = (\mathfrak{R}\phi)(x)+\sqrt{-1}(\mathfrak{I}\phi)(x)$ then $(\mathfrak{I}\phi)(x) = -\sqrt{-1}(\mathfrak{R}\phi)(\sqrt{-1}x)$. That is $\phi(x) = (\mathfrak{R}\phi)(x) - \sqrt{-1}(\mathfrak{I}\phi)(\sqrt{-1}x), \forall x \in E$. Therefore ϕ is continuous iff $\mathfrak{R}\phi$ is continuous. □

Definition 2.4.2 (Dual of a TVS)**.** Given a TVS E, the collection of continuous linear functionals on E is denoted by E^* and is called the dual of E.

2.5 Generality could be dull

Now we will show what is wrong with some of the examples.

Proposition 2.5.1. Consider the probability space $([0,1], \mathcal{B}_{[0,1]}, \lambda)$ be the Lebesgue measure space on the unit interval. Then for $0 < p < 1$, $(\mathcal{L}^p([0,1]))^* = \{0\}$. In other words these spaces do not admit any continuous linear functionals.

Proof. Suppose there exists a nonzero continuous linear functional ϕ. Since image of ϕ is a linear subspace it must be $\mathbb{K}$. So, there is some f so that $|\phi(f)| \geqslant 1$. Consider the continuous map

$$g : [0,1] \ni s \mapsto \int_0^s |f(t)|^p \, dt.$$

By the intermediate value theorem there exists s such that $g(s) = \frac{1}{2}\int_0^1 |f(t)|^p \, dt > 0$. Let $g_1 = f.\chi_{[0,s]}, g_2 = f.\chi_{(s,1]}$. We have $f = g_1 + g_2, |f|^p = |g_1|^p + |g_2|^p$. Therefore,

$$\int_0^1 |g_1(t)|^p \, dt = \frac{1}{2}\int_0^1 |f(t)|^p \, dt = \int_0^1 |g_2(t)|^p \, dt.$$

Since $|\phi(g_1) + \phi(g_2)| = |\phi(f)| \geqslant 1$, there exists i with $|\phi(g_i)| \geqslant \frac{1}{2}$. Let $f_1 = 2g_i$, so that $|\phi(f_1)| \geqslant 1$ and

$$\int_0^1 |f_1(t)|^p \, dt = 2^p \int_0^1 |g_i(t)|^p \, dt = 2^{p-1}\int_0^1 |f(t)|^p \, dt.$$

By iteration we get a sequence f_n such that

$$|\phi(f_n)| \geqslant 1, \int_0^1 |f_n(t)|^p \, dt = 2^{(p-1)n}\int_0^1 |f(t)|^p \, dt.$$

Note that $2^{p-1} < 1$, therefore $\{f_n\}$ is converging to zero. Then $\{\phi(f_n)\}$ must converge to 0 but this contradicts $|\phi(f_n)| \geqslant 1$. □

Now we must ask why is this happening? The following proposition gives a hint.

Proposition 2.5.2. Let E be a TVS and ϕ be a nonzero continuous linear functional on E, then there exists a proper convex neighbourhood of origin in E.

Proof. Let $V = \{x | |\phi(x)| < 1\}$. It is a proper convex neighbourhood of origin □

To understand the implications of this simple looking observation we need to explore convex neighbourhoods of origin further.

Theorem 2.5.3. *Let V be a convex neighbourhood of origin in a TVS E. Then there is an open, convex, balanced subset of V containing origin.*

Proof. Let V be a convex neighbourhood of origin. Then there exists a balanced open neighbourhood of origin $U \subseteq V$. Let W be the convex hull of U. Please recall it is defined as follows

$$W = \left\{ \sum_{i=1}^{n} \lambda_i x_i | n \in \mathbb{N}, \lambda_1, \ldots, \lambda_n \in [0,1], \sum_i \lambda_i = 1, x_i \in U, \forall i \right\}.$$

Since V is convex and $U \subseteq V$, we have $W \subseteq V$. Also $U \subseteq \mathring{W}$, the interior of W

The interior of W is convex. Let $x, y \in \mathring{W}$. For any $\lambda \in [0,1]$ we have to show that $z = \lambda x + (1-\lambda)y \in \mathring{W}$, or equivalently there exists an open neighbourhood of origin V' satisfying $z + V' \subseteq W$. That follows once we observe that there is an open neighbourhood of origin V' satisfying $x + V', y + V' \subseteq W$. For example we could take $V' = V_1 \cap V_2$ where V_1, V_2 are open neighbourhoods of origin with $x + V_1, y + V_2 \subseteq W$. □

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$\mathring{W}$ is balanced. The set W is balanced because the convex hull of a balanced set is balanced. Now let $x \in \mathring{W}$. Then $0.x = 0 \in U \subseteq \mathring{W}$. Let $0 \neq z \in \mathbb{K}$ with $|z| \leqslant 1$. Get a balanced neighbourhood of origin W' with $x + W' \subseteq W$. Then $z.x + z.W' \subseteq W$ because W is balanced. Therefore $z.x \in \mathring{W}$. □

So given any neighbourhood of origin V we have exhibited an open, convex, balanced neighbourhood of origin $\mathring{W} \subseteq V$. □

2.6 The Minkowski functional

Here is an an alternative way of describing absorbing balanced convex sets which turns out to be quite useful.

Definition 2.6.1. The Minkowski functional of an absobing set A is defined by
$$p_A = \inf\{t > 0 : t^{-1}x \in A\}.$$

Theorem 2.6.2. *Let A be a convex absorbing subset of a vector space E and p_A its Minkowski functional. Then*

1. *p_A is subadditive, i.e., $p_A(x+y) \leqslant p_A(x) + p_A(y), \forall x, y \in E$.*
2. *p_A is positively homogeneous, i.e., $p(\lambda x) = \lambda p(x), \forall \lambda \in \mathbb{R}_{>0}, x \in E$.*
3. *Moreover if A is balanced then p_A also satisfies $p_A(\lambda x) = |\lambda| p_A(x), \forall \lambda \in \mathbb{K}, x \in E$.*
4. *If E is a topological vector space and A is open then $A = \{x \in E : p_A(x) < 1\}$.*

Proof. (1)For all $\epsilon > 0$ we have λ, μ such that $p_A(x) \leqslant \lambda < p_A(x) + \epsilon, p_A(y) \leqslant \mu < p_A(y) + \epsilon$ and $\frac{x}{\lambda}, \frac{y}{\mu} \in A$. The convexity of A implies
$$\frac{x+y}{\lambda+\mu} = \frac{\lambda}{\lambda+\mu}\frac{x}{\lambda} + \frac{\mu}{\lambda+\mu}\frac{y}{\mu} \in A.$$
Therefore $p_A(x+y) \leqslant \lambda + \mu < p_A x) + p_A(y) + 2\epsilon$. Since ϵ is arbitrarily small, we obtain subadditivity.

(2), (3) Easily follows from the definition.

(4) Let $x \in A$. There exists an open neighborhood V of origin such that $x + V \subseteq A$. Since scalar multiplication is continuous there exists $\epsilon > 0$ such that $\epsilon.x \in V$. Then $(1+\epsilon)x \in A$. Therefore $p_A(x) \leqslant (1+\epsilon)^{-1} < 1$. Conversely suppose that $x \in E$ satisfies $p_A(x) < 1$. Then there exists $\epsilon \geqslant 0$ such that $\frac{x}{p_A(x)+\epsilon} \in A$ and $p_A(x) + \epsilon < 1$. Exploiting the convexity of A we get $x = (p_A(x)+\epsilon)\frac{x}{p_A(x)+\epsilon} + (1 - p_A(x) - \epsilon).0 \in A$. □

Definition 2.6.3. A real valued sub-additive function p defined on a vector space E is called a seminorm if $p(\alpha \cdot x) = |\alpha| p(x), \forall \alpha \in \mathbb{K}, x \in E$.

In this terminology theorem 2.6.2 can be restated as follows.

Theorem 2.6.4. *Let A be a convex absorbing balanced subset of a vector space E and p_A be its Minkowski functional. Then p_A is a seminorm. Moreover if E is a TVS and A is open then $A = \{x \in E : p_A(x) < 1\}$.*

The converse is also true. Before that we state a simple lemma.

Lemma 2.6.5. *Let* p *be a seminorm on a vector space* E*, then (a)* $p(0) = 0$*; (b)* $|p(x) - p(y)| \leqslant p(x - y), \forall x, y \in E$*; (c)* $p(x) \geqslant 0$*.*

Proof. (a) This follows from, $p(0) = p(0 \cdot x) = |0| \cdot p(x) = 0$.
(b) Note that

$$p(x) - p(y) = p(x - y + y) - p(y) \leqslant p(x - y) + p(y) - p(y) = p(x - y).$$

Interchanging x and y we obtain the other inequality $p(y) - p(x) \leqslant p(x - y)$ needed to complete the proof.
(c) We have $p(x) = p(x - 0) \geqslant |p(x) - p(0)| = |p(x)| \geqslant 0$. □

Theorem 2.6.6. *Let* p *be a seminorm on a vector space* E*. Then* $A = \{x : p(x) < 1\}$ *is a convex, balanced, absorbing set and* $p = p_A$*.*

Proof. Only thing we need to verify is $p = p_A$. If $x \in E$ and $s > p(x)$ then $s^{-1}x \in A$. Therefore $p_A(x) \leqslant p(x)$. On the other hand if $0 < t \leqslant p(x)$, then $t^{-1}x \notin A$. Hence $p(x) \leqslant p_A(x)$. □

Definition 2.6.7. Let p be a seminorm on a vector space E. Then $B_p = \{x \in E : \rho(x) < 1\}$ is called the unit open semiball or just semiball associated with the seminorm p.

In view of these results we will explore implications of the existence of a continuous linear functional.

Theorem 2.6.8. *Let* E *be a TVS and* ϕ *be a nonzero continuous linear functional. Then there exists a seminorm* p *such that there is a constant* $C > 0$ *with* $|\phi(x)| \leqslant Cp(x), \forall x \in E$*.*

Proof. We know that $V := \{x \| \phi(x)| < 1\}$ is a proper convex subset of E containing origin. By theorem 2.5.3 we get a convex, balanced, open neighbourhood of origin U contained in V. Let p be the Minkowski functional of U. Then $U = \{x | p(x) < 1\}$ and $|\phi(x)| < 1, \forall x \in U$.

Claim: If $p(x) = 0$ for some $x \in E$, then $\phi(x) = 0$.

Proof of claim. Since $p(x) = 0, \frac{x}{\epsilon} \in U, \forall \epsilon > 0$. Therefore $|\phi(\frac{x}{\epsilon})| < 1$ or $|\phi(x)| < \epsilon, \forall \epsilon > 0$. □

Let x be arbitrary. Then we have two possibilities. If $p(x) > 0$, then $\frac{x}{2p(x)} \in U$. Therefore $|\phi(\frac{x}{2p(x)})| < 1$ or equivalently $|\phi(x)| < 2p(x)$. If $p(x) = 0$, then $\phi(x) = 0$ as well, therefore we also have $|\phi(x)| \leqslant 2p(x)$. So, in either case we have $|\phi(x)| \leqslant 2p(x)$. $\square$

Chapter 3

Hahn-Banach theorems

Hahn-Banach theorems are of two kinds. We have Hahn-Banach extension theorems and Hahn-Banach separation theorems. These theorems are equivalent in the sense that one could have first proved separation theorems and then use them to deduce the extension theorems or one could do the other way. Separation theorems have geometric interpretations while extension theorems are analytic in nature.

3.1 Hahn-Banach extension theorems

Theorem 3.1.1 (Hahn Banach extension theorem first version)**.** *Let E be a real vector space and $p : E \longrightarrow \mathbb{R}$ a positively homogeneous subadditive function. Let $F \subseteq E$ be a subspace and $\phi : F \to \mathbb{R}$ a linear map satisfying $\phi(x) \leqslant p(x), \forall x \in F$. Then ϕ admits an extension $\tilde{\phi}$ to E satisfying $\tilde{\phi}(x) \leqslant p(x), \forall x \in E$.*

Proof. Step 1: Let $F_1 = F + \mathbb{R}x_0$, where $x_0 \in E \backslash F$. Let us denote a prospective candidate for $\tilde{\phi}(x_0)$ by ϕ_0. Then we must have

$$\phi(x) + \lambda\phi_0 \leqslant p(x + \lambda x_0), \forall x \in F, \lambda \in \mathbb{R}. \tag{3.1}$$

Considering the cases $\lambda \lesseqgtr 0$ in (3.1) we get

$$\phi(y) - p(y - x_0) \leqslant \phi_0 \leqslant p(x + x_0) - \phi(x), \forall x, y \in F \tag{3.2}$$

To show 3.2 we must show that $\sup_{y \in F}(\phi(y) - p(y - x_0)) \leqslant \inf_{x \in F}(p(x + x_0) - \phi(x))$ or equivalently

$$\phi(y) - p(y - x_0) \leqslant p(x + x_0) - \phi(x), \forall x, y \in F. \tag{3.3}$$

But this follows from $\phi(x)+\phi(y) = \phi(x+y) \leqslant p(x+y) \leqslant p(x+x_0)+p(y-x_0)$ because p satisfies triangle inequality and we can take any element from the interval $[\sup_{y \in F}(\phi(y) - p(y - x_0)), \inf_{x \in F}(p(x + x_0) - \phi(x))]$ as ϕ_0. Thus we have established the existence of an extension ϕ_1 of ϕ to F_1. Also from (3.1) we conclude that $\phi_1(x) \leqslant p(x), \forall x \in F_1$.

Step 2: Let $\mathcal{P} = \{(F_1, \phi_1) : F \subseteq F_1, \phi_1 \in F_1^*, \phi_1|_F = \phi, \phi_1(x) \leqslant p(x), \forall x \in F_1\}$. This is a POset with partial order given by $(F_1', \phi_1') \succeq (F_1, \phi_1)$. Every chain in $\mathcal{P}$ has an upper bound and therefore by Zorn's lemma $\mathcal{P}$ has a maximal element, say $(\tilde{F}, \tilde{\phi})$. We claim that $\tilde{F}$ must be E else by applying step 1 to $\tilde{F}$ we can obtain a further extension contradicting the maximality. □

Theorem 3.1.2 (Hahn Banach extension theorem second version)**.** *Suppose F is a subspace of a vector space E, p is a seminorm on E and $\phi : F \to \mathbb{K}$ a linear map such that $|\phi(x)| \leqslant p(x), \forall x \in F$. Then there is a linear functional $\tilde{\phi}$ defined on E such that $\tilde{\phi}|_F = \phi$ and $|\tilde{\phi}(x)| \leqslant p(x)$.*

Proof. Case 1 ($\mathbb{K} = \mathbb{R}$): We have $p(-x) = p(x)$ and we are done by theorem (3.1.1).

Case 2 ($\mathbb{K} = \mathbb{C}$): Let $\phi_1 = \Re\phi$, then there exists real linear $\tilde{\phi_1}$ on F such that $\tilde{\phi_1}|_E = \phi_1$ Let $\tilde{\phi}(x) = \tilde{\phi_1}(x) - i\tilde{\phi_1}(ix)$, then $\tilde{\phi}|_E = \phi$. Finally given any $x \in F, \exists \lambda \in \mathbb{C}$ such that $|\lambda| = 1, \lambda\tilde{\phi}(x) = |\tilde{\phi}(x)|$. We have,

$$|\tilde{\phi}(x)| = \tilde{\phi}(\lambda x) = \tilde{\phi_1}(\lambda x) \leqslant p(\lambda x) = p(x).$$ □

Now we can show that the converse of proposition 2.5.2 holds.

Proposition 3.1.3. Let E be a TVS admitting a convex neighbourhood of origin other than the whole space. Then there is a nonzero continuous linear functional.

Proof. Let V be a convex neighbourhood of origin other than the whole space. Then by theorem 2.5.3 we get a convex balanced open neighbourhood of origin A contained in V. Let p be the Minkowski functional of

A. Since $A = \{x|p(x) < 1\}$ and A is a proper subset there is x_0 so that $p(x_0) \geqslant 1$. Define a linear functional ϕ on $F = \mathbb{K}x_0$ by $\phi(x_0) = p(x_0)$. Then $|\phi(x)| \leqslant p(x), \forall x \in F$. By theorem 3.1.2 we get a linear functional $\tilde{\phi}$ such that $|\tilde{\phi}(x)| \leqslant p(x), \forall x \in E$ and $\tilde{\phi}(x_0) = \phi(x_0) \neq 0$. Also note that for all $x \in A, |\tilde{\phi}(x)| \leqslant p(x) < 1$. Therefore $\tilde{\phi}(A)$ is bounded and consequently by theorem 2.4.1 $\tilde{\phi}$ is continuous. □

3.2 Hahn-Banach separation theorems

Now we will discuss Hahn-Banach separation theorems. Because of their geometric interpretations these theorems are also called geometric forms of hahn-Banach theorems.

Theorem 3.2.1. *Let E be a topological vector space over $\mathbb{R}$ and A be a convex open neighborhood of the origin. Let $x_0 \notin A$, then there is a hyperplane separating x_0 from A, in other words there is a continuous linear functional $\ell \in E^*$ such that*

$$\ell(x_0) = 1 \text{ and } \ell(x) < 1, \quad \forall x \in A.$$

Proof. In a TVS scalar multiplication is continuous and A contains the origin. Therefore given any $x \in E$, the sequence x/n converges to 0, hence eventually enters the open neighborhood A. This shows that A is absorbing. Let p_A be the Minkowski functional of A. Then by theorem (**??**) we know that p_A is subadditive, positively homogeneous and $A = \{x \in E : p_A(x) < 1\}$. Since $x_0 \notin A$, we have $p_A(x_0) \geqslant 1$. On the one dimensional space spanned by x_0 define $\ell(\lambda x_0) = \lambda$. Then for $\lambda > 0$, $\ell(\lambda x) = \lambda \leqslant p_A(\lambda x_0)$. If $\lambda \leqslant 0$, then $\ell(\lambda x_0) = \lambda \leqslant 0 \leqslant p_A(\lambda x_0)$. In any case for any x from the subspace spanned by x_0 we have $\ell(x) \leqslant p_A(x)$. By theorem (3.1.1) we can extend ℓ to a linear map denoted by the same symbol ℓ on E such that $\ell(x) \leqslant p_A(x), \forall x \in E$. Then ℓ is continuous because if $x \in (-A) \cap A$, then $-1 < \ell(x) < 1$. □

Theorem 3.2.2. *Suppose A and B are disjoint nonempty convex sets in a topological vector space E. If A is open there exists $\phi \in E^*$ and $\gamma \in \mathbb{R}$ such that*

$$\Re\phi(x) < \gamma \leqslant \Re\phi(y), \forall x \in A, \forall y \in B.$$

If the scalar field is $\mathbb{R}$ then $\Re\phi := \phi$.

Proof. We will first do the case where the scalar field is $\mathbb{R}$. Fix $a_0 \in A$ and $b_0 \in B$. Put $x_0 = b_0 - a_0$ and $C = A - B + x_0$. Then C is open because it is a union of open sets $A - b + x_0, b \in B$. Clearly C is convex and contains the origin. Also $x_0 \in C$, because A and B are disjoint. Using theorem (3.2.1) obtain a continuous linear functional ϕ such that $\phi(x_0) = 1$ and $\phi(x) < 1, \forall x \in C$. If $a \in A, b \in B$, then $\phi(a - b + x_0) = \phi(a) - \phi(b) + 1 < 1$. Therefore, $\phi(a) < \phi(b)$. Let $\gamma = \inf\{\phi(b) : b \in B\}$. Then $\phi(a) \leqslant \gamma, \forall a \in A$. Since A is open we must have $\phi(a) < \gamma, \forall a \in A$.

If the scalar field is $\mathbb{C}$, there is a continuous real linear map ϕ_1 satisfies the assertion. If ϕ is the associated complex linear map whose real part is ϕ_1, then $\phi \in E^*$ and does the job. □

Corollary 3.2.3. Let B be a closed and convex subset of a locally convex space E and $x_0 \notin B$ then there exists $\phi \in E^*$ such that $\Re\phi(x_0) < \inf\{\Re\phi(x) : x \in B\}$.

Proof. Let A be a convex neighborhood of x_0 disjoint from B. Now apply theorem (3.2.2) □

Lemma 3.2.4 (Topological lemma). *Let E be a topological vector space, $C \subseteq E$ be a compact set and $D \subseteq E$ be a closed set. Then $C + D$ is closed.*

Proof. Since you are familiar with nets we will prove this using nets. Let $\{x_\alpha + y_\alpha\}_{\alpha \in A} \subseteq C + D$ be a convergent net with $\lim_\alpha(x_\alpha + y_\alpha) = z$. Since C is compact there exists a subnet $\{x_\beta\}$ converging to some $x \in C$. Then $\lim_\beta y_\beta = \lim_\beta(x_\beta + y_\beta - x_\beta) = z - x \in D$. So, we have $z = x + y \in C + D$. □

Theorem 3.2.5. *Let E be a locally convex space. Suppose $A, B \subseteq E$ are convex sets with A compact and B closed, $A \cap B = \emptyset$. Then there exists a linear continuous map $\phi : E \to \mathbb{K}$ and $\alpha, \beta \in \mathbb{R}$ such that*

$$\Re\phi(x) \leqslant \alpha < \beta \leqslant \Re\phi(y), \forall x \in A, \forall y \in B.$$

Proof. Consider the convex set $C = B - A$. By the topological lemma C is closed and $0 \notin C$, because $A \cap B = \emptyset$. Since E is locally convex there exists a convex open $D \subseteq E \setminus C$ containing the origin. In particular $C \cap D = \emptyset$. By theorem (3.2.2) we get a continuous linear map $\phi \in E^*$ and $\gamma \in \mathbb{R}$ such that

$$\Re\phi(d) < \gamma \leqslant \Re\phi(c), \forall d \in D, \forall c \in C.$$

Since $0 \in D, \gamma > 0$. The inequality $\Re\phi(c) \geqslant \gamma, \forall c \in C$ gives $\Re\phi(b) - \Re\phi(a) \geqslant \gamma > 0, \forall b \in B, \forall a \in A$. Let $\beta = \inf_{b \in B} \Re\phi(b), \alpha = \sup_{a \in A} \Re\phi(a)$. Then $\beta \geqslant \alpha + \gamma$ and we are done. □

3.3 Locally convex spaces, a convenient class of topological vector spaces

Since existence of proper convex neighbourhoods tantamount to existence of continuous linear functionals the next definition looks natural.

Definition 3.3.1. A topological vector space is said to be locally convex if there is a neighbourhood base at origin consisting of convex sets. A locally convex topological vector space will be referred as LCS or LCTVS.

Theorem 3.3.2. *An LCS has a basis of neighbourhoods of origin consisting of open, convex, balanced subsets.*

Proof. Follows from theorem 2.5.3. □

Given the correspondence between convex, absorbing, balanced sets and semiballs the following result is obvious.

Theorem 3.3.3. *Let E be a locally convex space. Then there exists a collection of seminorms $\{p_\alpha : \alpha \in A\}$ such that the associated semiballs give a fundamental system of neighbourhoods of origin. Conversely given a collection of seminorms $\{p_\alpha | \alpha \in A\}$ there exists a unique locally convex topology such that the associated semiballs generate a fundamental system of neighbourhoods of origin.*

Proof. We only need to argue the converse direction and that follows from the following theorem. □

Theorem 3.3.4. *Let E be a vector space over $\mathbb{K}$ and $\mathfrak{C}$ be a collection of absorbing, convex, balanced subsets of E. Then there exists a unique vector space topology on E turning it into a locally convex space so that the collection $\mathfrak{B}$ consisting of finite intersections of elements of $\mathfrak{U} := \{r.C | r > 0, C \in \mathfrak{C}\}$ forms a filter base generating the filter of neighbourhoods of origin.*

Proof. We have to employ corollary 2.1.13.

(i) Since elements of $\mathfrak{C}$ are balanced they contain origin. Therefore so does elements of $\mathfrak{B}$.

(ii) Intersections of balanced and absorbing sets are balanced and absorbing.

(iii) Only thing we need to is that given any $V \in \mathfrak{B}$ there exists $W \in \mathfrak{B}$ such that $W + W \subseteq V$. Note that V is convex because intersections of convex sets are convex. Also $\frac{1}{2}V \in \mathfrak{B}$. So we can take $W = \frac{1}{2}V$. □

3.4 Examples of locally convex spaces

So far our discussion is quite abstract in the sense that we haven't discussed much of examples. Only example we have talked about is that of L^p spaces and that was used to illustrate a pathological property of lack of continuous linear functionals. Now we will address that. We will begin with examples of locally convex spaces. We have already seen that locally convex topologies on $\mathbb{K}$-vector spaces can be specified in terms of semi norms. So, it is enoiugh to produce seminorms. This does not look difficult.

Example 3.4.1. Let E be a vector space and $\phi : E \to \mathbb{K}$ be a linear functional. Then $p_\phi : E \ni x \mapsto |\phi(x)| \in \mathbb{R}$ is a seminorm.

Definition 3.4.2. Let E be a $\mathbb{K}$-vector space and it's algebraic dual be $E' \coloneqq \{\phi : E \to \mathbb{K} | \phi \text{ is a linear functional}\}$. Given any subspace $\mathcal{A} \subseteq E'$, we use $\sigma(E, \mathcal{A})$ to denote the locally convex topology on the vector space E prescribed by the collection of seminorms $\{p_\phi : \phi \in \mathcal{A}\}$.

Two instances of this is most useful.

Definition 3.4.3 (Weak* topology)**.** If E is a locally convex space then using the canonical embedding of E inside E^{**} we can consider E as a subspace of E^{**} and the topology $\sigma(E^*; E)$ is called the weak-* topology on E^*. A net $\{\phi_\lambda\} \subseteq E^*$ converges in the weak* topology to ϕ iff $\lim_\lambda \phi_\lambda(x) = \phi(x), \forall x \in E$.

Definition 3.4.4 (Weak topology)**.** Let E be a locally convex space. Then $\sigma(E;E^*)$ is called the weak topology on E. A net $\{x_\lambda\} \subseteq E$ converges to x if $\lim_\lambda \phi(x_\lambda) = \phi(x), \forall \phi \in E^*$.

Definition 3.4.5. Let E be an LCS. For any set $A \subseteq E$, it's (right) polar A° is defined by

$$A^\circ := \{\phi \in E^* | \sup_{x\in A} |\phi(x)| \leqslant 1\}.$$

Similarly, for any set $A \subseteq E^*$ we define it's (left/pre) polar ${}^\circ A$ by

$${}^\circ A := \{x \in E : \sup_{\phi\in A} |\phi(x)| \leqslant 1\}.$$

Theorem 3.4.6 (Banach-Alaoglu-Bourbaki)**.** *Let E be an LCTVS and A be a convex, balanced, neighbourhood of origin of E. Then A° is compact in the weak*-topology.*

Proof. For $x \in E$, let $S_x := \{z \in \mathbb{K} | |z| \leqslant p(x)\}$ where p is the Minkowski functional associated with A. Consider $S := \prod_{x\in E} S_x$ with the product topology. Let $x \in E$ and $\phi \in A^\circ$. Then for $\epsilon > 0, x \in (p(x)+\epsilon).A$. Therefore $|\phi(x)| \leqslant p(x)+\epsilon$. Since ϵ is arbitrary, $|\phi(x)| \leqslant p(x)$ or equivalently $\phi(x) \in S_x$. Therefore we can define $\Phi : A^\circ \to S$ by $\Phi(\phi)_x = \phi(x)$. Obviously Φ is one to one and allows us to identify A° as a subset of S. This identification respects topology. This means $\Phi : (A^\circ, weak*) \to \Phi(A^\circ)$ is a homeomorphism. To see this observe that if $\{\phi_\lambda\}$ is a net in A°, then $\phi_\lambda \to \phi \in A^\circ$ in weak*-topology iff $\Phi(\phi_\lambda) \to \Phi(\phi)$ in S. By Tychonov's theorem S is compact and it is clearly Hausdorff. Therefore to show A° is compact it suffices to show that $\Phi(A^\circ)$ is compact. Which in turn follows once we show that it is closed. Let $\{\phi_\lambda\} \subseteq A^\circ$ be a net and $\Phi(\phi_\lambda) = \psi_\lambda \in S$. Suppose $\psi_\lambda \to \psi \in S$. In other words $\lim \psi_{\lambda_x} = \psi_x, \forall x \in E$. We have to show that there exists $\phi \in A^\circ$ such that $\Phi(\phi) = \psi$. That will follow once we show that the association $\phi : E \ni x \mapsto \psi_x$ is linear and $\sup_{x\in A} |\phi(x)| \leqslant 1$. Let us show them one by one begining with linearity. Let $a, b \in \mathbb{K}, x, y \in E$,

$$\begin{aligned}\psi_{(ax+by)} &= \lim \psi_{\lambda(ax+by)} = \lim \phi_\lambda(ax+by) = \lim a\phi_\lambda(x) + \lim b\phi_\lambda(y) \\ &= \lim a\psi_{\lambda_x} + \lim b\psi_{\lambda_y} = a\psi_x + b\psi_y.\end{aligned}$$

To see $\phi \in A^\circ$ observe that for all $x \in A$, $|\phi(x)| = \lim |\phi_\lambda(x)| \leqslant 1$. So, $\psi = \Phi(\phi)$. This establishes $\Phi(A^\circ)$ is closed and consequently it is compact. □

Example 3.4.7 (Smooth functions)**.** Let $\Omega \subseteq \mathbb{R}^d$ be an open subset. Consider the vector space $C^k(\Omega)$ of k-times continuously differentiable functions on Ω. For each compact subset $K \subseteq \Omega$ and $m \in \mathbb{N}_0$ (recall we use $\mathbb{N}_0$ to denote $\{0\} \cup \mathbb{N}$) consider the seminorm

$$p_{K,m}(f) = \sum_{|\alpha| \leqslant m} \sup_{x \in K} |\frac{\partial^{\alpha_1 + \cdots + \alpha_d}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}} f(x)|.$$

Then the locally convex space given by $C^k(\Omega)$ along with the family of seminorms $\{p_{K,k} | K \subseteq \Omega \text{ is compact }\}$ is denoted by $\mathfrak{C}^k(\Omega)$. Convergence in $\mathfrak{C}^k(\Omega)$ means uniform convergence on compact subsets for all derivatives upto order k. We use $\mathfrak{C}(\Omega)$ to denote the locally convex space obtained by considering $C^\infty(\Omega)$ along with the family $\{p_{K,m} : m \in \mathbb{N}, K \subseteq \Omega \text{ is compact }\}$.

Example 3.4.8 (Schwartz space)**.** Let $\mathcal{S}(\mathbb{R})$ be the space of smooth functions on $\mathbb{R}^d$ with rapidly decaying derivatives. In other words

$$\mathcal{S}(\mathbb{R}) = \{f : \mathbb{R}^d \to \mathbb{K} : \forall k, m \in \mathbb{N}_0, \sup_{x \in \mathbb{R}^d, |\alpha| \leqslant m} (1 + \|x\|_2)^k |D^\alpha f(x)| < \infty\},$$

where $\|x\|_2 := \sqrt{\sum_{j=1}^d x_j^2}$. This is called the Schwartz space. Equipped with the family of seminorms

$$p_{k,m}(f) := \sup_{x \in \mathbb{R}^d, |\alpha| \leqslant m} (1 + \|x\|_2)^k |D^\alpha f(x)|, k, m \in \mathbb{N}_0,$$

Schwartz space is a locally convex space. It's dual is the space of tempered distributions. Obviously we could have defined Schwartz space $\mathcal{S}(V)$ for any finite dimensional real vector space V. Fourier transform is a continuous isomorphism from $\mathcal{S}(V)$ to $\mathcal{S}(V^*)$.

Definition 3.4.9 (Strict inductive limit)**.** Let E be a vector space and $\{E_\alpha | \alpha \in A\}$ be a collection of subspaces with $E = \cup E_\alpha$. Suppose each $(E_\alpha, \mathcal{T}_\alpha)$ is a locally convex topological vector space. They are compatible in the sense that if $E_{\alpha_1} \subseteq E_{\alpha_2}$ then the topology of E_{α_1} coincides with the relative topology inherited from E_{α_2}. Let

$$\mathfrak{C} := \{U \subseteq E | U \text{ is convex, balanced, absorbing }, U \cap E_\alpha \in \mathcal{N}_\alpha\},$$

where $\mathcal{N}_\alpha$ is the neighbourhood filter at origin of E_α. To see that this is a rich collection for each α take a convex, balanced neighbourhood of origin U_α. Let U be the convex hull of $\cup_\alpha U_\alpha$. Then U is a convex, balanced and absorbing subset so that for each α, $U \cap E_\alpha$ is a convex balanced neighbourhood of origin in E_α. By theorem 3.3.4 we get a locally convex vector space topology $\mathcal{T}$ on E called the strict inductive limit topology of E_α's. We use the notation $(E, \mathcal{T}) = \text{s-}\lim(E_\alpha, \mathcal{T}_\alpha)$ to denote $(E, \mathcal{T})$ is the strict inductive limit of $(E_\alpha, \mathcal{T}_\alpha)$'s.

Proposition 3.4.10. Let $(E, \mathcal{T}) = \text{s-}\lim(E_\alpha, \mathcal{T}_\alpha)$. Then $\mathcal{T}_\alpha$ contains the relative topology of E_α inherited as a subspace of the topological space E.

Proof. Let V_α be a neighbourhood of origin in the relative topology of E_α. Then $V_\alpha = V \cap E_\alpha$ where V is a neighbourhood of origin in E. We know that

$$V \supseteq \cap_{j=1}^{k} \epsilon_j U_j \text{ for some } U_j \in \mathfrak{C}; \epsilon_j > 0, j = 1, \ldots, k.$$

Therefore $V_\alpha \supseteq \cap_{j=1}^{k} \epsilon_j (U_j \cap E_\alpha)$. This shows that V_α is a neighbourhood in the topology of E_α. □

Theorem 3.4.11. *Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of locally convex spaces with $E_n \subseteq E_{n+1}, \forall n$ and the topology of E_n is the relative topology inherited from E_{n+1} for all n. Suppose $E = \cup E_n$ and $E = \text{s-}\lim E_n$. Then the relative topology of E_n as a subspace of E coincides with the original topology. It is for this reason we use the adjective strict.*

Lemma 3.4.12. *Let X be a locally convex TVS, X_0 a linear subspace equipped with the subspace topology, and U a convex (balanced) neighbourhood of the origin in X_0. Then there exists a convex (balanced) neighbourhood V of the origin in X such that $V \cap X_0 = U$.*

Proof. There exists a neighbourhood W of origin in X such that $U = W \cap X_0$. Since X is locally convex there is a convex(balanced) neighbourhood W_0 of origin in X such that $W_0 \subseteq W$. Let V be the convex hull of $U \cup W_0$. Then by construction V is a convex neighbourhood of origin in X and $U \subseteq V$. Therefore $U = U \cap X_0 \subseteq V \cap X_0$. Let $x \in V \cap X_0$. As U and W_0 are convex we may write $x = \lambda y + (1 - \lambda) z$ with $y \in U, z \in W_0$ and $\lambda \in [0, 1]$. If $\lambda = 1$, then $x = y \in U$. If $0 \leqslant \lambda < 1$ then $z = (1 - \lambda)^{-1}(x - \lambda y) \in X_0$. So, $z \in W_0 \cap X_0 \subseteq W \cap X_0 = U$. This implies by convexity of $U, x \in U$. Hence $V \cap X_0 = U$. □

Proof of theorem. Let $n \in \mathbb{N}$. Only thing we need to show is that the topology of E_n is coarser than the relative topology. Let U_n be a convex balanced neighbourhood of origin in the topology of E_n. By the lemma we obtain U_{n+1}, a convex balanced neighbourhood of origin in E_{n+1} so that $U_{n+1} \cap E_n = U_n$. By induction get a convex balanced neighbourhood U_{n+k} of the origin in E_{n+k} such that $U_{n+k} \cap E_{n+k-1} = U_{n+k-1}$. Hence for any k we get $U_{n+k} \cap E_n = U_n$. Let $U = \cup_k U_{n+k}$. Then U is a convex, balanced neighbourhood of origin in E with $U \cap E_n = U_n$. Thus U_n is open in the relative topology. □

Example 3.4.13 (Test functions $\mathcal{D}(\Omega)$)**.** Let $\Omega \subseteq \mathbb{R}^d$ be open and $C_c^\infty(\Omega)$ be the collection of compactly supported smooth functions with support contained in Ω. For each compact $K \subseteq \Omega$, let $C^\infty(K)$ be the collection of smooth functions with support contained in K. Let us fix a collection of compact sets $\{K_n\}_{n\in\mathbb{N}}$ so that $\mathring{K}_n \supseteq K_{n-1}, K_0 = \emptyset, \forall n$ and $\Omega = \cup K_n$. Equipped with the seminorms $\{p_{K,m} | m \in \mathbb{N} \cup \{0\}\}$, $C^\infty(K)$ becomes a locally convex topological vector space. Note that $C^\infty(K_n) \subseteq C^\infty(K_{n+1}), \forall n$ and the topology of $C^\infty(K_n)$ coincides with the subspace topology. We also have $C_c^\infty(\Omega) = \cup C^\infty(K_n)$ and $C_c^\infty(\Omega)$ with the strict inductive limit topology of $\{C^\infty(K_n)\}$ is denoted by $\mathcal{D}(\Omega)$. This topology is also referred as LF topology meaning limits of Frechet spaces.

Proposition 3.4.14. A sequence $\{f_k\}_k$ converges to 0 in $\mathcal{D}(\Omega)$only if (i) there exists a compact set $K \subseteq \Omega$ so that $supp(f_k) \subseteq K, \forall k$ and (ii) for each multiindex α, $\{D^\alpha f_k\}$ converges to 0 uniformly on K.

Proof. Only thing we need to show is (i). Suppose (i) does not hold. Then there exists a sequence $\{x_k\}$ without a convergent subsequence and a subsequence $\{f_{k_n}\}$ with $f_{k_n}(x_n) \neq 0$. Then the seminorm

$$p(f) = \sum_{n=1}^{\infty} 2 \sup_{x \in (K_n \setminus K_{n-1})} |f(x)|/|f_{k_n}(x_n)|$$

where the sequence of compact sets $\{K_n\}$ satisfies $\mathring{K}_n \supseteq K_{n-1}, K_0 = \emptyset, \forall n, \Omega = \cup K_n$, and $x_n \in K_n \setminus K_{n-1}$ defines a neighbourhood of origin $U = \{f | p(f) < 1\}$. None of the f_{k_n}'s belong to U. This contradicts $\lim f_k = 0$. □

Remark 3.4.15. From the inclusion $C_c^\infty(\Omega) \subseteq C^\infty(\Omega)$, $C_c^\infty(\Omega)$ gets a subspace topology. But $\mathcal{D}(\Omega)$ is a strictly finer topology. To show this we need to exhibit a sequence $\{f_n\}$ converging to zero in the relative topology but not the LF topology. Choose $f \in \mathcal{D}(\mathbb{R})$ with support $[-1, 1]$ and let $f_n(x) = f(x)/n$. Then $f_n \in \mathcal{D}(\mathbb{R})$. and

$$\|f_n\|_\infty := \sup_{x \in \mathbb{R}} |f_n(x)| = \frac{1}{n}\|f\|_\infty \to 0.$$

Also,

$$\|f_n^{(k)}\}_\infty := \sup_{x \in \mathbb{R}} \frac{1}{n^{k+1}} |f^{(k)}(x/n)| = \frac{1}{n^{k+1}} \|f^{(k)}\|_\infty \to 0.$$

Therefore $f_n \to 0$, in the topology of $C^\infty(\mathbb{R})$ but $\{f_n\}$ does not converge in $\mathcal{D}(\Omega)$ because $\text{supp}(f_n) = [-n, n]$ is growing arbitrarily large.

Definition 3.4.16. A seminorm p on a vector space E is called a norm if $p(x) = 0$ implies $x = 0$. Norms are often denoted by $\|\cdot\|$. A normed linear space $(E, \|\cdot\|)$ is a vector space E equipped with a norm $\|\cdot\|$.

Practice problems

1. (On compactly supported smooth functions) Purpose of this exercise is to give you some idea about compactly supported smooth functions. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$f(t) \begin{cases} = 0, & \text{for } t \leqslant 0; \\ > 0 & \text{for } t > 0. \end{cases}$$

For example we could take $f(t) = e^{-1/t}$ for $t > 0$ and $f(t) = 0$ for $t \leqslant 0$. Define

$$f_0 : \mathbb{R}^d \ni x \mapsto f(1 - \|x\|_2^2) \in \mathbb{R}.$$

(i) Show that f_0 is compactly supported and infinitely differentiable.

(ii) Show that if $f \in C_c^\infty(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)$ are compactly supported, then

$$(f \star g)(x) := \int_{\mathbb{R}^d} f(x-y)g(y)dy,$$

is also in $C_c^\infty(\mathbb{R}^d)$. (Hint: Show that $\frac{\partial (f\star g)}{\partial x_i}(x) = (\frac{\partial f}{\partial x_i} \star g)(x)$.

(iii) For $\epsilon > 0$, let $f_\epsilon(x) = C(\epsilon) f_0(\frac{x}{\epsilon})$ where $C(\epsilon) = \left(\int f_0(\frac{x}{\epsilon})dx\right)^{-1}$. Let $h \in C_c^k(\mathbb{R}^d), 0 \leqslant k < \infty$. Define $h_\epsilon = h \star f_\epsilon$. Show that

$$\operatorname{supp}(h_\epsilon) \subseteq \{x | \exists y \in \operatorname{supp}(h), \|x - y\|_2 \leqslant \epsilon\} =: K_\epsilon.$$

Also show that $D^\alpha h_\epsilon \to D^\alpha h$ uniformly on $\mathbb{R}^d$ for all multiindex α with $|\alpha| \leqslant k$.

(iv) Show that $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for $1 \leqslant p < \infty$.

2. Let $f \in C_c^\infty(\mathbb{R})$, then show that the limit $\lim_{\epsilon \to 0} \left(\int_{-\infty}^{-x} \frac{f(t)}{t} dt + \int_x^\infty \frac{f(t)}{t} dt \right)$ exists. This limit is denoted by $P.V\left[\frac{1}{x}\right](f)$.

3. Let E be a TVS and F be a subspace. Show that with the quotient topology E/F is a TVS and this is Hausdorff iff F is closed.

3.5 Assignment II, due on 11/02/25

Throughout these exercises $\Omega \subseteq \mathbb{R}^d$ stands for an open set.

1. Let E, F be locally convex spaces with topologies prescribed by families of seminorms $\mathfrak{P}, \mathfrak{Q}$ respectively. For a linear map $T : E \to F$ the following are equivalent.

   (a) T is continuous.

   (b) T is continuous at zero.

   (c) For all $q \in \mathfrak{Q}$ there exists $n \in \mathbb{N}, p_1, \cdots, p_n \in \mathfrak{P}, C > 0$ such that $q(T(x)) \leqslant C \max_i p_i(x)$.

2. Let $\{E_n\}$ be a sequence of locally convex spaces and E be their strict LF limit. Let F be a locally convex space. Show that a linear map $\phi : E \to F$ is continuous iff $\phi|_{E_n}$ is continuous for all n.

3. Show that the inclusion map $C_c^\infty(\Omega)$ to $L^p(\Omega)$ is continuous for $0 < p < \infty$.

4. Show that a linear functional $\phi : \mathcal{D}(\Omega) \to \mathbb{K}$ is continuous iff $\phi(f_j) \to 0$ for all sequences $\{f_j\} \subseteq \mathcal{D}(\Omega)$ converging to zero.

5. Let $\phi \in (\mathcal{D}(\Omega))'$. Define $(\partial_i\phi)(f) \coloneqq \phi(-\partial_i f)$. Show that $\partial_i\phi$ is a distribution. Also show that $\partial_i : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ is continuous.

6. Let $f \in L^1(\mathbb{R})$. Show that $\mathcal{D}(\mathbb{R}) \ni g \mapsto f \star g \in \mathcal{D}(\mathbb{R})$ is continuous.

7. Let $L^1_{Loc}(\Omega)$ be the space of locally integrable functions defined by

$$L^1_{Loc}(\Omega) \coloneqq \{f : \Omega \to \mathbb{K} | f \text{ is measurable and for all compact } K \subseteq \Omega, \int_K |f| < \infty.\}$$

   Show that for all $f \in L^1_{Loc}(\Omega)$ the map $\mathcal{D}(\Omega) \ni g \mapsto \int f(x)g(x)dx$ is a distribution denoted by the same symbol.

8. Show that $H : \mathcal{D}(\mathbb{R}) \ni f \mapsto \int_0^\infty f(x)dx$ is a distribution called Heaviside distribution.

9. Let $f \in C_c^\infty(\Omega)$. Now we have two interpretations of the symbol $\partial_i f$. We can differentiate as a distribution and consider the resulting distribution or we can differentiate as a function and consider the associated distribution. Show that both these distributions are same. This explains the negative sign in problem 5.

10. For each $a \in \Omega$, show that the map $\delta_a : \mathcal{D}(\Omega) \ni f \mapsto f(a)$ is a distribution called the Dirac distribution at a.

11. Show that the distributional derivative $\frac{dH}{dx} = \delta_0$.

12. Show that P.V. $\left[\frac{1}{x}\right] : \mathcal{D}(\Omega) \ni f \mapsto \lim_{x\to 0}\left(\int_{-\infty}^{-x} \frac{f(t)}{t}dt + \int_x^\infty \frac{f(t)}{t}dt\right)$ is a distribution, called the principal value of $\frac{1}{x}$.

13. Show that the map $\mathbb{R} \ni x \mapsto \log(|x|$ is in $L^1_{Loc}(\mathbb{R})$. Also show that $\frac{d\log(|x|)}{dx} = $ P.V. $\left[\frac{1}{x}\right]$.

3.6 Normed spaces as examples of LCS

Here is our definition of normed linear space. Since we are discussing locally convex spaces our definition looks a bit convoluted but we will immediately argue that this definition is same as the usual definition.

Definition 3.6.1. A Hausdorff locally convex topological vector space with topology specified by a single seminorm is called a normed linear space.

Proposition 3.6.2. Let p be a seminorm on a vector space E. Then the topology generated by p is Hausdorff iff $p(x) = 0$ only if $x = 0$.

Proof. Let $p(x) = 0$ and $x \neq 0$. Then $x \in V, \forall V \in \mathcal{N}_0$. But this contradicts the Hausdorff hypothesis. □

Definition 3.6.3. A seminorm $p : E \to \mathbb{R}$ is said to be a norm if $p(x) = 0$ only if $x = 0$. Often norms are denoted by $\| \cdot \|$.

Therefore we can also define normed linear spaces as topological vector spaces with topology specified by a norm. We will use the notation $(E, \| \cdot \|_E)$ to denote a normed linear space E, that comes equipped with a norm $\| \cdot \|_E$. If there is no scope of confusion and the subscript E in $\| \cdot \|_E$ appears a bit notationally overwhelming we may drop it from notation. These are metric spaces with the metric associated with the norm $\| \cdot \|$ given by $d_{\| \cdot \|}(x, y) = \|x - y\|$.

Theorem 3.6.4. *Let $F \subseteq E$ be normed linear spaces and $\phi : F \to \mathbb{K}$ be a continuous linear functional then there exists a continuous linear functional $\tilde{\phi} : E \to \mathbb{K}$ so that $\tilde{\phi}|_F = \phi$ and $\|\tilde{\phi}\| = \|\phi\|$. Such a $\tilde{\phi}$ is called a norm preserving extension of ϕ.*

Proof. Consider the seminorm $p(x) = \|\phi\|\|x\|$. Then we have $|\phi(x)| \leqslant p(x), \forall x \in F$. By theorem 3.1.2 we obtain a linear functional $\tilde{\phi}$ on E so that $\tilde{\phi}|_F = \phi$ and $|\tilde{\phi}(x)| \leqslant p(x) = \|\phi\|\|x\|, \forall x \in E$. Therefore $\|\tilde{\phi}\| \leqslant \|\phi\|$. Since $\tilde{\phi}$ extends ϕ we obviously have the other inequality required to show $\|\tilde{\phi}\| = \|\phi\|$. □

Corollary 3.6.5 (Corollary to Hahn-Banach Theorem)**.** Let E be a normed linear space and $x \in E$. Then there exists $x^* \in E^*$ such that $x^*(x) = \|x\|, \|x^*\| = 1$.

Proof. Let F be the span of x and $\phi : F \to \mathbb{K}$ be the linear functional given by $\phi(\lambda x) = \lambda\|x\|, \forall \lambda \in \mathbb{K}$. Then $\|\phi\| = 1$. Let x^* be a norm preserving extension of ϕ. □

Corollary 3.6.6 (Corollary to Hahn-Banach Theorem)**.** Let E be a normed linear space and E^* it's dual. Then the norm of $x \in E$ satisfies,

$$\|x\| = \sup\{| < x^*, x > | : \|x^*\| \leqslant 1\},$$

where $< x^*, x >$ denotes $x^*(x)$.

Proof. Let $x \in E$, then for any $x^* \in E^*$ with $\|x^*\| \leqslant 1$, we have $| < x^*, x > | \leqslant \|x^*\|\|x\| \leqslant \|x\|$. This shows that

$$\|x\| \leqslant \sup\{| < x^*, x > | : \|x^*\| \leqslant 1\}.$$

For the other inequality using the Hahn Banach theorem obtain x^* of norm one such that $x^*(x) = \|x\|$. □

Now that we have shown that E^* is a nontrivial space it makes sense to recognise one crucial property enjoyed by duals of normed linear spaces, namely completeness. Stefan Banach initiated systematic study of these spaces and he called them B spaces. Frechet started calling them Banach spaces. Let us officially record the definition.

Definition 3.6.7 (Banach Space)**.** A complete normed linear space is called a Banach space

Proposition 3.6.8. Let E be a normed linear space and F be a Banach space. Then $\mathcal{L}(E, F)$ is a Banach space. In particular E^* is a Banach space.

Proof. Let $\{T_n\}$ be a Cauchy sequence in $\mathcal{L}(E, F)$. Then $\forall \epsilon > 0, \exists N$ such that $\|T_n - T_m\| < \epsilon, \forall n, m \geqslant N$. Then for any $x \in E$,

$$\|T_n x - T_m x\| < \epsilon\|x\| \text{ for } n, m \geqslant N. \tag{3.4}$$

Using completeness of F we get $\lim T_n x = Tx$. Also

$$T(\alpha x + \beta y) = \lim T_n(\alpha x + \beta y) = \lim \alpha T_n(x) + \beta T_n(y) = \alpha T(x) + \beta T(y).$$

Therefore T is linear and it is bounded because

$$\|T(x)\| = \lim \|T_n(x)\| = \lim \|T_N(x) + (T_n(x) - T_N(x))\| \leqslant (\epsilon + \|T_N\|)\|x\|.$$

Letting m tend to infinity in (3.4) we get $\|T_n - T\| \leqslant \epsilon, \forall n > N$. Thus $T = \lim T_n \in \mathcal{L}(E, F)$ showing completeness of $\mathcal{L}(E, F)$. □

Proposition 3.6.9. Let E be a Banach space. A subspace $F \subseteq E$ is complete iff it is closed.

Proof. If part: Let $\{x_n\} \subseteq F$ be a Cauchy sequence. Then using completeness of E we know $\lim x_n = x$ for some $x \in E$. Since F is closed $\lim x_n = x \in F$. Thus F is complete.

Only if part: Let $\{x_n\} \subseteq F$ be converging to x. As F is complete $x \in F$. Therefore F is closed. □

Exercise 3.6.10. Show that a finite dimensional subspace of a normed linear space is always closed. Hint: Any two norms on a finite dimensional space are equivalent.

Chapter 4

Applications of Hahn Banach theorems

We will begin with few applications of the Hahn-Banach extension theorem.

4.1 Canonical embedding into second dual

Definition 4.1.1. Let $j_E : E \to E^{**}$ be the map defined by $j_E(x)(x^*) = < x^*, x >$. Then

$$\|j_E(x)\| = \sup_{x^* : \|x^*\| = 1} | < x^*, x > | = \|x\|.$$

Therefore j_E is an isometric embedding of E into E^{**}, often referred as the canonical embedding of E into E^{**}. The norm closure of $j_E(E)$ is the completion of E. We say E is reflexive if j is an isomorphism.

Proposition 4.1.2. Let E be a normed linear space. Then the completion of E is a Banach space.

Proof. The norm closure of $j_E(E)$ is the completion of E. Being closure of a subspace it is a complete normed linear space or which is same as a Banach space. □

Remark 4.1.3. Can there be a non-reflexive normed linear space E such that there is an isometric isomorphism $T \in \mathcal{L}(E, E^{**})$, i.e., an isomorphism T

satisfying $\|T(x)\| = \|x\|, \forall x \in E$? A counter example was given by Robert James. It is in his honour we denote the canonical embedding by j.

Definition/Proposition 4.1.4. Let E, F be Banach spaces and $T \in L(E, F)$. Then $T^* : F^* \to E^*$ defined by $T^*(\phi)(x) = (\phi \circ T)(x)$ defines a bounded linear map, called the adjoint of T with $\|T^*\| = \|T\|$. Also $I_E^* = I_{E^*}$, where I_E, I_{E^*} be the identity mappings of E, E^* respectively. If $S \in \mathcal{L}(F, G)$ then $(S \circ T)^* = T^* \circ S^*$.

Proof. Let $\phi \in F^*$ then

$$\begin{aligned} \|T^*(\phi)\| &= \sup\{|T^*(\phi)(x)| : x \in E, \|x\| \leqslant 1\} \\ &= \sup\{|\phi(T(x))| : x \in E, \|x\| \leqslant 1\} \\ &\leqslant \|\phi\|\|T\|. \end{aligned}$$

Therefore $\|T^*\| \leqslant \|T\|$. We give two proofs of the other inequality $\|T\| \leqslant \|T^*\|$.

First proof.

$$\begin{aligned} \|T\| &= \sup\{\|T(x)\| : x \in E, \|x\| \leqslant 1\} \\ &= \sup\{|\phi(T(x))| : x \in E, \phi \in F^*, \|x\|, \|\phi\| \leqslant 1\} \\ &\leqslant \sup\{\|T^*(\phi)\| : \phi \in F^*, \|\phi\| \leqslant 1\} \\ &\leqslant \|T^*\|. \end{aligned}$$

□

Second proof. Let $x \in E, \phi \in F^*$. Then we have

$$T^{**}(j_E(x))(\phi) = j_E(x)(T^*\phi) = T^*(\phi)(x) = \phi(T(x)) = j_F(T(x))(\phi).$$

In other words

$$T^{**} \circ j_E = j_F \circ T. \tag{4.1}$$

In categorical parlance this means j is a natural transformation. (Soon we will elaborate on this.) Therefore,

$$\|T\| = \sup_{x \in B_E} \|T(x)\| = \sup_{x \in B_E} \|j(T(x))\| = \sup_{x \in B_E} \|T^{**}(j(x))\| \leqslant \sup_{x^{**} \in B_{E^{**}}} \|T^{**}(x^{**})\| = \|T^{**}\|$$

Using $\|T^*\| \leqslant \|T\|$ for T^* we get $\|T^{**}\| \leqslant \|T^*\|$. Thus $\|T\| \leqslant \|T^*\|$. □

[Lecture Notes of P.S.Chakraborty]

Let us look back and reflect on what have we done just now. To any normed linear space E we have associated a normed linear space, namely E^*. Also to any $T \in \mathcal{L}(E, F)$ we have associated a $T^* \in \mathcal{L}(F^*, E^*)$. This association satisfies two more properties, (i) $I_E^* = I_{E^*}$ and (ii) $S \in \mathcal{L}(F, G)$ then $(S \circ T)^* = T^* \circ G^*$. Now in mathematics whenever some structure occurs frequently we introduce terminology so that we can talk about the structure and investigate its properties. In this case the relevant structure is of categories and functors.

4.2 Categories and functors

Definition 4.2.1 (Locally small category)**.** A locally small category $\mathcal{C}$ consists of a class $Ob(\mathcal{C})$ called objects of $\mathcal{C}$ and given any two objects $A, B \in Ob(\mathcal{C})$, a set $Mor_{\mathcal{C}}(A, B)$ called morphisms of $\mathcal{C}$. When there is no scope for confusion we will drop $\mathcal{C}$ from the notation $Mor_{\mathcal{C}}$. If $f \in Mor(A, B)$, then we may also write $f : A \to B$ or $A \xrightarrow{f} B$. We will denote $Mor(A, A)$ by $Mor(A)$. Given $A, B, C \in Ob(\mathcal{C})$, there is a map $\circ : Mor(A, B) \times Mor(B, C) \to Mor(A, C)$ called composition and for each $A \in Ob(\mathcal{C})$ a morphism $I_A \in Mor(A)$, called the identity morphism of A such that $\forall f \in Mor(A, B), g \in Mor(B, C), \forall h \in Mor(C, D)$ we have $\circ(\circ(f, g), h) = \circ(f, \circ(g, h))$ and $\circ(I_A, f) = f = \circ(f, I_B)$. We denote $\circ(f, g)$ by $g \circ f$. In this notation the conditions become associativity $h \circ (g \circ f) = (h \circ g) \circ f$ and $f \circ I_A = f = I_B \circ f$.

Example 4.2.2. The category $\mathcal{S}ets$ has sets as objects and functions as morphisms.

Example 4.2.3. The category $\mathcal{G}p$ has groups as objects and group homomorphisms as morphisms. The usual composition of functions define composition.

Example 4.2.4. Let G be a group. Then we can define a category with only one object $*$ and $Mor(*) = G$. The identity element of G plays the role of I_* while the group multiplication defines the composition. This example shows morphisms may not be functions. Also in a sense the notion of category generalises the notion of groups.

Example 4.2.5. The category $\mathcal{N}ls_{\mathbb{K}}$ the category of normed linear spaces over $\mathbb{K}$ has normed $\mathbb{K}$ vector spaces as objects and bounded linear maps as mrphisms.

Example 4.2.6. The category $\mathcal{B}an$ has Banach spaces as objects with $Mor(E,F) = \mathcal{L}(E,F)$.

Example 4.2.7. The category $\mathcal{B}an_1$ has Banach spaces as objects with $Mor(E,F) = \{T \in \mathcal{L}(E,F) : \|T\| \leqslant 1\}$.

Definition 4.2.8. Let $\mathcal{C}, \mathcal{D}$ be categories. A covariant (contravariant) functor $F : \mathcal{C} \to \mathcal{D}$ associates to an object $A \in Ob(\mathcal{C})$ an object $F(A) \in Ob(\mathcal{D})$ and to a morphism $f \in Mor_{\mathcal{C}}(A,B)$ an element $F(f) \in Mor_{\mathcal{D}}(F(A), F(B))(F(f) \in Mor_{\mathcal{D}}(F(B), F(A))$ such that

1. For all f, g so that the composition $g \circ f$ is defined we have $F(g) \circ F(f) = F(g \circ f)(F(f) \circ F(g) = F(g \circ f))$.
2. For all $A \in Ob(\mathcal{C}), F(I_A) = I_{F(A)}$.

Covariant functors are often called functors.

In this terminology we can state what we have already proved.

Example 4.2.9. The dualization functor $* : \mathcal{N}ls_{\mathbb{K}} \to \mathcal{N}ls_{\mathbb{K}}$ is the contravariant functor sending $E \in Ob(\mathcal{N}ls_{\mathbb{K}})$ to E^* and $T \in \mathcal{L}(E,F)$ to T^*. Since dualization is contravariant applying it twice we get the covariant functor second dual.

Definition 4.2.10. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. Then a natural transformation $\eta : F \to G$ associates a morphism $\eta_A \in Mor_{\mathcal{D}}(F(A), G(A))$ for each object A of $\mathcal{C}$ so that for each $f \in Mor_{\mathcal{C}}(A,B)$ we have $\eta_B \circ F(f) = G(f) \circ \eta_A$. This is also expressed by saying the following diagram commutes.

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\eta_B} & G(B) \end{array}$$

Example 4.2.11. The James map gives a natural transformation $j : Id \to **$. We have verified the relevant condition in (4.1).

4.3 Reflexive Banach spaces

Proposition 4.3.1. A closed subspace of a reflexive Banach space is reflexive.

Proof. Let $F \subseteq E$ be a closed subspace with $i : F \hookrightarrow E$ the inclusion map. Let $y^{**} \in F^{**}$. We have to exhibit $y \in F$ such that $j_F(y) = y^{**}$. Since E is reflexive there is $x \in E$ such that $i^{**}(y^{**}) = j_E(x)$. It is enough to show that $x \in F$. In other words $i(x) = x$. Because then $i^{**}(y^{**}) = j_E(x) = j_E \circ i(x) = i^{**}(j_F(x))$. If we can show i^{**} is one to one then we will get $y^{**} = j_F(x)$. So we need to show two things, (i) $x \in F$ and i^{**} is one to one.

Proof of $x \in F$. Suppose $x \notin F$. Then by Hahn-Banach there exists $x^* \in E^*$ such that $x^*(F) = 0$ or equivalently $i^*(x^*) = 0$ and $x^*(x) = 1$. We have the following chain of equalities

$$1 = \langle x^*, x\rangle = \langle j_E(x), x^*\rangle = \langle i^{**}(y^{**}), x^*\rangle = \langle y^{**}, i^*(x^*)\rangle = 0!$$

This contradiction shows $x \in F$. □

Injectivity of i^{**}.Let $y^* \in F^*$ be arbitrary and x^* be a norm preserving extension of y^*, in other words $\langle x^*, i(y)\rangle = \langle y^*, y\rangle, \forall y \in F$. So, $\langle i^*(x^*) - y^*, y\rangle = \langle x^*, i(y)\rangle - \langle y^*, y\rangle = 0, \forall y \in F$. Thus $y^* = i^*(x^*)$. In other words i^* is onto. Suppose $i^{**}(z^{**}) = 0$ for some $z^{**} \in F^{**}$. Then for all $x^* \in E^*$ we have $\langle z^{**}, i^*(x^*)\rangle = 0$. Since i^* is onto, this means $z^{**} = 0$ □

Proposition 4.3.2. Let E be a Banach space. Then E is reflexive iff E^* is reflexive.

Proof. Only if part: Let E be reflexive. We have to show every $x^{***} \in E^{***}$ is of the form $j_{E^*}(x^*)$. So, given x^{***} define x^* by

$$\langle x^*, x\rangle = \langle x^{***}, j_E(x)\rangle, \forall x \in E. \tag{4.2}$$

Claim: $j_{E^*}(x^*) = x^{***}$.

Proof of claim. We have to show $\langle x^{***}, x^{**}\rangle = \langle j_{E^*}(x^*), x^{**}\rangle, \forall x^{**} \in E^{**}$. So, let $x^{**} \in E^{**}$ be arbitrary. Then using reflexivity of E we get $x^{**} = j_E(x)$ for some $x \in E$. The following chain of equalities

$$\langle j_{E^*}(x^*), x^{**}\rangle = \langle x^{**}, x^*\rangle = \langle j_E(x), x^*\rangle = \langle x^{***}, j_E(x)\rangle = \langle x^{***}, x^{**}\rangle$$

show $x^{***} = j_{E^*}(x^*)$. □

If part: If E^* is reflexive then by the only if part E^{**} is reflexive. By proposition (4.3.1), $j_E(E)$ is reflexive. Therefore so is E. □

Proposition 4.3.3. Let E, F be isomorphic Banach spaces. Then E is reflexive iff F is reflexive

Proof. It is enough to show one of the implications because the other follows by symmetry. We will show the only if part. Let $T : E \to F$ be an isomorphism. Then $T^{**} : E^{**} \to F^{**}$ is an isomorphism. Since James map is a natural transformation we have $T^{**} \circ j_E = j_F \circ T$. The left hand side is surjective because E is reflexive. Therefore the right hand side must be surjective as well. Since T is an isomorphism this implies j_F is surjective. □

4.4 Hilbert spaces

Definition 4.4.1. A Banach space is said to be a Hilbert space if it's norm is associated with an inner product.

Exercise 4.4.2. Let $(E, \|\cdot\|)$ be a Banach space. Then E is a Hilbert space iff $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2), \forall x, y \in E$. This identity is calledthe parallelogram identity.

Proposition 4.4.3. Let E be a finite dimensional Banach space. Then any two norms on E are equivalent.

Corollary 4.4.4. Any finite dimensional subspace of a Banach space is closed.

Proposition 4.4.5. Let E be a locally compact Banach space then E must be finite dimensional.

Exercise 4.4.6. Let $\mathcal{H}_1 \subseteq \mathcal{H}_2$ be finite dimensional Hilbert spaces. Show that any linear functional on $\mathcal{H}_2$ admits a unique norm preserving extension to $\mathcal{H}_1$.

[Lecture Notes of P.S.Chakraborty]

4.5 Duals of some Banach spaces

We identify duals of some standard Banach spaces and their dual norms. We begin with the simplest.

Proposition 4.5.1. Let ℓ_n^p denote $\mathbb{K}^n$ equipped with the norm $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $1 \leqslant p < \infty$ and $\|x\|_p = \max_i |x_i|$ for $p = \infty$. Let $q \in (1, \infty]$ be the conjugate exponent of p. This means q satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual of ℓ_n^p is isometrically isomorphic with ℓ_n^q.

Proof. Define $\Phi : (\mathbb{K}^n, \|\cdot\|_q) \to (\mathbb{K}^n, \|\cdot\|_p)^*$ by $\Phi(y)(x) = \sum_{i=1}^n x_i y_i$. Then by Holder's inequality $|\Phi(y)(x)| \leqslant \|y\|_q \|x\|_p, \forall x, y$. Therefore $\|\phi(y)\| \leqslant \|y\|_q$. To show the oppsite inequality $\|\Phi(y)\| \geqslant \|y\|_q$ we consider the following cases separately (i) $1 < p < \infty$ and $p = 1$. For $u \in \mathbb{K}$ define $sign(u) = |u|/u$ if $u \neq 0$ and $sign(u) = 0$ otherwise. If $p = 1$ then the conjugate exponent is ∞. Let i_0 be such that $|y_{i_0}| = \max_i |y_i| = \|y\|_\infty$. Consider the vector $x \in \mathbb{K}^n$ with $x_i = 0$ for $i \neq i_0$ and $x_{i_0} = sign(y_{i_0})$. Then $\|x\|_1 = 1$ and $|\Phi(y)(x)| = |y_{i_0}| = \|y\|_\infty$. This shows that $\|\Phi(y)\| \geqslant \|y\|_\infty$. If $p > 1$, then given $y \in \mathbb{K}^n$ consider the vector x with $x_i = sign(y_i)|y_i|^{q-1}$. Then $\|x\|_p = \|y\|_q^{q/p}$ and $|\Phi(y)(x)| = \|y\|_q^q$. Therefore $\|\Phi(y)\| \geqslant \frac{|\Phi(y)(x)|}{\|x\|_p} = \|y\|_q$. Since $\|\Phi(y)\| = \|y\|, \forall y$ the map Φ is one to one. The domain and codomain of Φ both being n-dimensional Φ must be onto. In other words Φ is an isometric isomorphism. □

Proposition 4.5.2. Let $\ell^p := \{x \in \mathbb{K}^\infty : \sum_i |x_i|^p < \infty\}$ for $1 \leqslant p < \infty$. We have already seen that ℓ^p is a banach space equipped with the norm $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$. The dual of ℓ^p is isometrically isomorphic with ℓ^q.

Proof. □

Remark 4.5.3. Next proposition requires Radon-Nikodym theorem. If you are familiar with the result for complex measures then you can consider the case $\mathbb{K} = \mathbb{C}$. Else you have to consider the case of real scalars only.

Proposition 4.5.4. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $1 \leqslant q < \infty$. Suppose p is the conjugate exponent of q. Define $\Phi : L^p(\Omega, \mathcal{A}, P) \to (L^q(\Omega, \mathcal{A}, P))^*$ by

$$\Phi(f)(g) = \int f.g dP.$$

Then Φ is an isometric isomorphism for $1 < p \leqslant \infty$.

Proof. By Holder's inequality $\|\Phi(f)\| \leqslant \|f\|_p$. We will first show that Φ is an isometry and then we will show Φ is onto. To show that Φ is an isometry as before the argument splits into two cases.

Let $p = \infty, q = 1$. If $\|f\|_\infty = 0$, then there is nothing to show. Let $\|f\|_\infty > \epsilon > 0$. Consider the function

$$g_\epsilon(x) = \frac{sign(f(x))\chi_{\{\omega \in \Omega : |f(\omega)| > \|f\|_\infty - \epsilon\}}(x)}{P(\{\omega \in \Omega : |f(\omega)| > \|f\|_\infty - \epsilon\})}.$$

Here for a set $A \in \mathcal{A}, \chi_A$ denotes the indicator function of A. Then $g_\epsilon \in L^1$ and $\|g_\epsilon\|_1 = 1$. Note that $\Phi(f)(g_\epsilon) = \int fg_\epsilon \geqslant \|f\|_\infty - \epsilon$. Since ϵ could be arbitrarily small $\|\Phi(f)\| \geqslant \|f\|_\infty$.

If $p < \infty$ or equivalently $1 < q$, given $f \in L^p$ let us take

$$g(x) = sign(f(x))|f(x)|^{p-1}$$

Then $\|g\|_q^q = \|f\|_p^p$ and $\Phi(f)(g) = \|f\|_p^p$. This shows that $\|\Phi(f)\| \geqslant \|f\|_p$.

Only thing remains to be shown is surjectivity of Φ. For that we adopt the following strategy, given a continuous linear functional $\phi \in (L^q(\Omega, \mathcal{A}, P))^*$ we first produce a measure ν_ϕ on $(\Omega, \mathcal{A})$ absolutely continuous with respect to P. Then the Radon-Nikodym derivative $\frac{d\nu_\phi}{dP}$ is absolutely integrable. We will show that it is actually in L^p and $\Phi(\frac{d\nu_\phi}{dP}) = \phi$.

Let $\phi \in (L^q(\Omega, \mathcal{A}, P))^*, 1 \leqslant q < \infty$. Define $\nu_\phi : \mathcal{A} \to \mathbb{K}$ as $\nu_\phi(A) = \phi(\chi_A)$. Then by linearity we conclude that ν_ϕ is finitely additive. Let $\{A_n\}_{n\in\mathbb{N}}$ be a countable collection of mutually disjoint measurable sets and $B_k = \cup_{n=1}^k A_n, k \in \mathbb{N}, B = \cup_{n=1}^\infty A_n$. Then $\|\chi_{B_k} - \chi_B\|_q \to 0$. Here we need $q < \infty$. Therefore using continuity of ϕ we conclude that $\nu_\phi(B_k) \to \nu_\phi(B)$. This shows countable additivity of ν_ϕ. In other words ν_ϕ is a complex measure in case $\mathbb{K} = \mathbb{C}$, else it is a signed measure. Clearly $\nu_\phi << P$. Let us denote the Radon-Nikodym derivative $\frac{d\nu_\phi}{dP}$ by f. Then $f \in L^p(\Omega, \mathcal{A}, P)$. We wish to show actually $f \in L^p(\Omega, \mathcal{A}, P)$ and $\Phi(f) = \phi$. Note that $\nu_\phi(A) = \phi(\chi_A) = \int f \cdot \chi_A dP$ for all $A \in \mathcal{A}$. By linearity we get for all simple functions g

$$\Phi(f)(g) := \int f \cdot g dP = \phi(g). \tag{4.3}$$

We wish to establish that $f \in L^p(\Omega, \mathcal{A}, P)$ and 4.3 for all $g \in L^q(\Omega, \mathcal{A}, P)$. We will establish this by dividing the problem in two cases.

Let $q = 1, p = \infty$. Given any $A \in \mathcal{A}$, let g_A be the simple function

$$g_A(x) = \frac{sign(f(x))\chi_A}{P(A)}.$$

Then $g_A \in L^1((\Omega, \mathcal{A}, P))$. Let $\{g_{A,n}\}$ be a sequence of simple functions converging to g_A pointwise and bounded in absolute value. Then by bounded convergence theorem the sequence $\{g_{A,n}\}$ converges to g_A in $L^1((\Omega, \mathcal{A}, P))$. Note that

$$\phi(g_A) = \lim \phi(g_{A,n}) = \lim \int f \cdot g_{A,n} dP = \int f \cdot g_A dP = \int_A \frac{|f|}{P(A)} dP.$$

Therefore,

$$\int_A \frac{|f|}{P(A)} dP \leqslant |\phi(g_A)| \leqslant \|\phi\| \|g_A\|_1 = \|\phi\|.$$

Since this is true for all measurable A, we must have $f \in L^\infty(\Omega, \mathcal{A}, P)$ and $\|f\|_\infty \leqslant \|\phi\|$. Now given an arbitrary $g \in L^1((\Omega, \mathcal{A}, P))$, let us obtain a sequence of simple functions converging to g pointwise and $|g_n(x)| \uparrow |g(x)|, \forall x \in \Omega$. Since we have already established that $f \in L^\infty((\Omega, \mathcal{A}, P))$, by dominated convergence theorem we conclude 4.3 for our g.

Let $1 < q < \infty, 1 < p < \infty$. Obtain a sequence of simple functions $\{f_n\}$ such that (i) $|f_n(x)| \uparrow |f(x)|, \forall x \in \Omega$ and (ii) $\lim f_n(x) = f(x), \forall x \in \Omega$. Take

$$g_n(x) = |f_n(x)|^{(p-1)} \frac{sign(f(x))}{\|f_n\|_p^{(p-1)}}.$$

Since $sign(f(x))$ is bounded in absolute value by 1, each g_n is a bounded function. Obtain a sequence of bounded simple functions $\{g_{nm}\}$ converging pointwise to g_n. Then by bounded convergence theorem g_{nm} converges to g_n in $L^q((\Omega, \mathcal{A}, P))$. Therefore we conclude that the relation 4.3 holds for g_n. In other words we have

$$\phi(g_n) = \int f \cdot g_n dP.$$

Also

$$\|g_n\|_q^q = \int \frac{|f_n|^{(p-1)q}}{\|f_n\|_p^{(p-1)q}} dP = 1.$$

Therefore,

$$\|\phi\| \geqslant |\phi(g_n)| = \int \frac{|f||f_n|^{(p-1)}}{\|f_n\|_p^{(p-1)}} dP \geqslant \|f_n\|_p.$$

So,

$$\|\phi\|\|\phi\|^{(p-1)} \geqslant \|\phi\|\|f_n\|_p^{(p-1)} \geqslant \int |f||f_n|^{(p-1)} dP.$$

By an application of Fatou's lemma we conclude that

$$\int |f|^p \leqslant \liminf \int |f||f_n|^{(p-1)} dP \leqslant \|\phi\|^p.$$

Now an application of Hölder's inequality proves 4.3 for all $g \in L^q((\Omega, \mathcal{A}, P))$.

□

4.6 Dual of $C_0(X)$, for a locally compact Hausdorff space X

We wish to identify the dual of the Banach space $C_0(X;\mathbb{C})$ of complex valued continuous functions on a locally compact space$X$ vanishing at infinity. For $X = [0,1]$ this was done by F. Riesz in 1909. A. Markov extended this to some noncompact spaces and the version in this generality is due to S. Kakutani. He obtained this in 1941. We will see the proof by Garling. Idea behind his proof is the cute observation that on a totally disconnected space it is easy to construct measures. Using the canonical continuous surjection from βX, the Stone-Cech compactification of X to X he obtains an embedding of $C(X)$ into $C(\beta X)$. Then an application of Hahn-Banach theorem allows him to extend the given linear functional to $C(\beta X)$. Now one appeals to the existence of clopen base in βX and obtains a Baire measure. Finally one appeals to extension theorems. The result is somewhat lengthy, so we need to follow the development carefully.

Definition 4.6.1. Let X be a compact Hausdorff space. A bounded linear functional $\rho \in (C(X))^*$ is said to be Hermitian if $\rho(f^*) = \overline{\rho(f)}$, where f^* is the function $x \mapsto \overline{f(x)}$. In other words for every self-adjoint f, i.e., f satisfying $f = f^*$ or equivalently for every real valued function f, we have $\rho(f) \in \mathbb{R}$.

Proposition 4.6.2. Let X be a compact Hausdorff space and $\rho \in C(X)^*$. Then there exists unique Hermitian functionals ρ_i, ρ_2 so that $\rho(f) = \rho_1(f) + i\rho_2(f), \forall f \in C(X)$.

Proof. If there are such ρ_1, ρ_2 then we will have $\overline{\rho(f^*)} = \overline{\rho_1(f)} + \overline{i\rho_2(f)}$. Therefore we must have $\rho_1(f) = \frac{1}{2}(\rho(f) + \overline{\rho(f^*)})$ and $\rho_2(f) = \frac{1}{2i}(\rho(f) - \overline{\rho(f^*)})$. We take these expressions as definitions of ρ_1, ρ_2 and verify that they are bounded Hermitian linear functionals on $C(X)$ satisfying $\rho(f) = \rho_1(f) + i\rho_2(f), \forall f \in C(X)$. Uniqueness is obvious from the construction. □

Definition 4.6.3. A function $f \in C(X; \mathbb{C})$ is said to be positive if f is real valued and $f(x) \geqslant 0, \forall x \in X$. This is denoted by $f \geqslant 0$. A Hermitian linear functional $\rho \in (C(X))^*$ is said to be positive if $\rho(f) \geqslant 0$ for all $f \geqslant 0$. This is denoted by $\rho \geqslant 0$. Given two bounded linear functionals ρ_1, ρ_2 we write $\rho_1 \geqslant \rho_2$ to say both ρ_1, ρ_2 are Hermitian and $\rho_1 - \rho_2 \geqslant 0$.

Proposition 4.6.4. Let $\rho \in (C(X; \mathbb{C})^*$ be a bounded Hermitian linear functional. Then there exists unique positive linear functionals $\rho_\pm$ so that $\rho = \rho_+ - \rho_-$ and if $\rho = \rho_1 - \rho_2$ is another such decomposition then we must have $\rho_+ \leqslant \rho_1, \rho_- \leqslant \rho_2$.

Proof. Let $f \geqslant 0$. Since ρ is Hermitian, for any $0 \leqslant g \leqslant f$, we know in particular g is real. Since ρ is Hermitian, for such g's $\rho(g) \in \mathbb{R}$ and $\rho(g) \leqslant |\rho(g)| \leqslant \|\rho\|\|g\| \leqslant \|\rho\|\|f\|$. That means the set $\{\rho(g) : 0 \leqslant g \leqslant f\}$ is bounded and we can legitimately define $\rho_+(f) = \sup\{\rho(g) : 0 \leqslant g \leqslant f\}$. Taking $g = 0, f$ we conclude that $\rho_+(f) \geqslant \max\{\rho(f), 0\}$. We define $\rho_-(f) = \rho_+(f) - \rho(f)$. Then $\rho_-(f) \geqslant 0$ for each $f \geqslant 0$. Now we will establish that we can extend ρ as a linear functional. That is done in steps.

Step 1: Claim : $\rho_+(f_1) + \rho_+(f_2) = \rho_+(f_1 + f_2), \forall f_1, f_2 \geqslant 0$:

Proof of claim. Let $0 \leqslant g_i \leqslant f_i, i = 1, 2$. Then $0 \leqslant g_1 + g_2 \leqslant f_1 + f_2$. Therefore $\rho_+(f_1) + \rho_+(f_2) \leqslant \rho_+(f_1 + f_2)$. To show equality we have to establish the other inequality. Let $\epsilon > 0$ and $0 \leqslant g \leqslant (f_1 + f_2)$ be such that $\rho_+(f_1 + f_2) \leqslant \rho(g) + \epsilon$. Define $g_1(x) = \min\{g(x), f_1(x)\}, \forall x \in X$. Then $g_1 \in C(X)$, satisfies $0 \leqslant g_1 \leqslant f_1$. Define $g_2 = g - g_1$. Then $0 \leqslant g_2$ and $g = g_1 + g_2$. We must have $g_2 \leqslant f_2$ as well because otherwise if for some $x, g_2(x) > f_2(x)$ then for such an x we must have

$$0 \leqslant f_2(x) < g_2(x) = g(x) - \min\{g(x), f_1(x)\}.$$

Thus $g(x) > \min\{g(x), f_1(x)\}$. Therefore $g(x) > f_1(x)$ and $g_1(x) = f_1(x)$. So, $g(x) = g_1(x) + g_2(x) = f_1(x) + g_2(x) > f_1(x) + f_2(x)$, a contradiction establishing $g_2 \leqslant f_2$. So,

$$\rho_+(f_1 + f_2) \leqslant \rho(g) + \epsilon = \rho(g_1) + \rho(g_2) + \epsilon \leqslant \rho_+(f_1) + \rho_+(f_2) + \epsilon.$$

Since ϵ is arbitrary we have the other inequality required to establish our claim. □

Step 2: Claim : For a self-adjoint element $f \in C(X)$ express f as $f = f_1 - f_2$ with $f_1, f_2 \geqslant 0$ and define $\rho_+(f) = \rho_+(f_1) - \rho_+(f_2)$. Of course we have to show this is well defined.

Proof of claim. Firstly we define $f_+ = \max\{f, 0\}$ and $f_- = f - f_+$. Then $f_\pm \in C(X)$ and satisfy $f = f_+ - f_-$. Indeed we can decompose every real valued continuous function as a difference of two nonnegative continuous functions. Now suppose $f = f_1 - f_2 = f_1' - f_2'$ be two such decompositions. Then $f_1 + f_2' = f_1' + f_2$. Therefore by step 1, $\rho_+(f_1) + \rho_+(f_2') = \rho_+(f_1') + \rho_+(f_2)$ and consequently

$$\rho_+(f) = \rho_+(f_1) - \rho_+(f_2) = \rho_+(f_1') - \rho_+(f_2')$$

becomes well defined. □

Step 3: Claim : For $0 < a \in \mathbb{R}$ and $0 \leqslant f \in C(X)$, $\rho_+(a \cdot f) = a \cdot \rho_+(f)$.

Proof pf claim. This is immediate once we note that

$$\begin{aligned} \rho_+(a \cdot f) &= \sup\{\rho(g) : 0 \leqslant g \leqslant a \cdot f\} \\ &= \sup\{\rho(a \cdot g) : 0 \leqslant g \leqslant f\} \\ &= a \cdot \sup\{\rho(g) : 0 \leqslant g \leqslant f\} \\ &= a \cdot \rho_+(f). \qquad \square \end{aligned}$$

Step 4: Claim : For all real valued continuous function f we have $\rho_+(-f) = -\rho_+(f)$.

Proof pf claim. If $f = f_+ - f_-$ where $f_\pm \geqslant 0$, then $-f = f_- - f_+$ and the claim follows. □

Now for an arbitrary continuous function if we define $\rho_+(f) = \rho_+(\Re(f)) + i\rho_+(\Im(f))$ where $\Re(f), \Im(f)$ are respectively the real and imaginary parts of the function f. Then ρ_+ is a positive linear functional. Clearly $\rho_- := \rho_+ - \rho$ is a linear functional and we have already noted that it is a positive linear functional. □

Here is a convenient way of deciding whether a linear functional is positive or not.

Proposition 4.6.5. Let $\rho : C(X) \to \mathbb{K}$ be a linear functional. We are not hypothesising boundedness of ρ. Then ρ is positive iff ρ is bounded with $\|\rho\| = \rho(I_X)$, where $I_X : X \to \mathbb{K}$ denotes the constant function $x \mapsto 1 \in \mathbb{K}$.

Proof. Only if part: Consider the semi-inner product on $C(X)$ given by $\langle f, g\rangle = \rho(f^*g)$. Then Cauchy-Schwarz inequality gives

$$|\langle f, g\rangle|^2 = |\rho(f^*g)|^2 \leqslant \rho(f^*f)\rho(g^*g) = \langle f, f\rangle\langle g, g\rangle.$$

Putting $f = I_X$ we get $|\rho(g)|^2 \leqslant \rho(I_X)\rho(|g|^2) \leqslant \|g\|^2\rho(I_X)^2$. The last inequality follows from $\rho(\|g\|^2 I_X - |g|^2) \geqslant 0$, a consequence of positivity. Therefore $\|\rho\| \leqslant \rho(I_X)$. Obviously $\rho(I_X) \leqslant \|\rho\|$ because $\|I_X\| = 1$. This completes proof of $\|\rho\| = \rho(I_X)$. If part: Without loss of generality we assume that $\rho(I_X) = \|\rho\| = 1$. It is enough to show that $0 \leqslant f \leqslant I_X$ implies $0 \leqslant \rho(f) \leqslant 1$. Suppose $\rho(f) = z \in \mathbb{C} \setminus [0, 1]$ for some f. Then we can find an open disc centred at z_0 with radius $r > 0$ which contains $[0, 1]$ but not z. Then for any $x \in X$, we have $|f(x) - z_0| < r$. Therefore $\|f - z_0 I_X\| < r$. Hence $|z - z_0| = |\rho(f - z_0 I_X)| \leqslant \|f - z_0 I_X\| < r$. This contradicts $|z - z_0| \geqslant r$. □

Proposition 4.6.6. Let ρ be a Hermitian linear functional. Then $\|\rho\| = \|\rho_+\| + \|\rho_-\|$.

Proof. By triangle inequality we get

$$\|\rho\| \leqslant \|\rho_+\| + \|\rho_-\| = \rho_+(I) - \rho_-(I) = 2\rho_+(I) - \rho(I)|.$$

For the other inequality note that if $0 \leqslant f \leqslant I$, then $-I \leqslant 2f - I \leqslant I$. So, $\|2f - I\| \leqslant 1$ and $2\rho_+(I) - \rho(I) = \sup\{\rho(2f - I) : 0 \leqslant f \leqslant I\} \leqslant \sup\{\|\rho\|\|(2f - I)\||0 \leqslant f \leqslant I\} \leqslant \|\rho\|$. □

Definition 4.6.7. Given a topological space X, the Baire σ-algebra $\mathcal{B}a_X$ is the smallest σ-algebra of subsets of X, so that every element of $C(X;\mathbb{R})$ becomes $\mathcal{B}a_X$ measurable.

Definition 4.6.8. Let X be a compact Hausdorff space. A measure μ on $\mathcal{B}_X$, the Borel σ-algebra of X is said to be regular if it satisfies the following conditions.

1. Outer regular on Borel sets:

$$\forall A \in \mathcal{B}_X, \mu(A) = \inf\{\mu(U) : A \subseteq U, U \text{ is open}\}.$$

2. Inner regularity on compact sets:

$$\forall \text{ open set } U \subseteq X, \mu(U) = \sup\{\mu(K) | K \subseteq U, K \text{ is compact}\}.$$

Theorem 4.6.9 (Riesz-Markov-Kakutani). *Let X be a compact Hausdorff topological space and $\rho \in (C(X;\mathbb{C})^*$ be a positive linear functional. Then there exists a unique regular finite measure μ on the Borel σ-algebra $\mathcal{B}_X$ such that $\rho(f) = \int f d\mu$.*

We will learn the proof by Garling. His prove has two very clear parts. There is a functional analytic part and a measure theoretic part. We will do the functional analytic part and for the measure theoretic part we will state it clearly with a clear reference.

Proof of Garling. Let βX be the Stone-Cech compactification when we consider X as a discrete set. Then βX is a compact Hausdorff space. Then using the universal property of Stone-Cech compactification there exists a unique map $\phi : \beta X \to X$ so that $\phi(\tau(x)) = x, \forall x \in X$, where $\tau : X \to \beta X$ is the canonical embedding. Let $C(\phi) : C(X;\mathbb{C}) \to C(\beta X;\mathbb{C})$ be the map $C(\phi)(f) = f \circ \phi$. Since φ is surjective, $C(\phi)$ is an injective isometry. Let $\tilde{\rho} = \rho \circ C(\phi)^{-1} \in (C(\phi)(C(X;\mathbb{C}))^*$. By the Hahn-Banach extension theorem $\tilde{\rho}$ admits a norm preserving extension denoted by the same symbol to a linear functional on $C(\beta X;\mathbb{C})$. Then $\tilde{\rho}(I_{\beta X}) = \rho \circ C(\phi)^{-1}(C(\phi)I_X) = \rho(I_X) = \|\rho\| = \|\tilde{\rho}\|$. By proposition 4.6.5, we conclude that $\tilde{\rho}$ is a positive linear functional on $C(\beta X;\mathbb{C})$. Let $\mathcal{A}$ be the Boolean algebra of subsets of

βX which are both closed and open. Then for each $A \in \mathcal{A}, \chi_A$ is a continuous function on βX and the span $\mathcal{A}lg$ of $\{\chi_A : A \in \mathcal{A}\}$ is an involutive, associative algebra of continuous functions on βX containing the constant functions that separates points of βX. Therefore by the Stone-Weirstrass theorem $\mathcal{A}lg$ is dense in $C(\beta X)$ in the norm topology. If we denote by $\sigma(\mathcal{A})$ the σ-algebra generated by $\mathcal{A}$, then each $\chi_A, A \in \mathcal{A}$ is $\sigma(\mathcal{A})$ measurable. In fact $\sigma(\mathcal{A})$ is the smallest σ-algebra with respect to which every element of $\mathcal{A}lg$ is measurable. Now using the density of $\mathcal{A}lg$ in $C(\beta X)$ we conclude that $\mathcal{B}a_{\beta X} = \sigma(\mathcal{A})$. Define $\tilde{\mu} : \mathcal{A} \to [0, \infty)$ as $\tilde{\mu}(A) = \tilde{\rho}(\chi_A)$. Positivity of $\tilde{\rho}$ implies $\tilde{\mu}$ takes values in $[0, \infty)$. By linearity of $\tilde{\rho}$, we get that $\tilde{\mu}$ is finitely additive. Let us show it's countable additivity. Let $A = \cup A_n$ be a countable union of disjoint elements of $\mathcal{A}$. Since A is a closed subset of a compact topological space it is compact. Each A_n is open, therefore by compactness of A there is a finite subcover. That means except finitely many n's rest of the A_n's are empty. So, countable additivity reduces to finite additivity. Thus $\tilde{\mu}$ is a premeasure with $\tilde{\mu}(\beta X) = \tilde{\rho}(\chi_{\beta X}) = \|\tilde{\rho}\|$. By Caratheodory's extension theorem $\tilde{\mu}$ admits a unique extension to a measure denoted by the same symbol $\tilde{\mu}$ on $\mathcal{B}a_{\beta X}$. Now it's time to invoke the measure theoretic input in the argument.

Theorem 4.6.10. *Let μ be a finite measure on the Baire σ-algebra of a compact Hausdorff space Y. Then μ admits an extension to a regular measure on the Borel σ-algebra $\mathcal{B}_Y$.*

This result is available in section 7.3 of Dudley, Real Analysis and Probability. Now using this result for the compact Hausdorff space βX we obtain a regular measure still denoted by the same symbol $\tilde{\mu}$ on $\mathcal{B}_{\beta X}$. For each $A \in \mathcal{A}$ we have $\int \chi_A d\tilde{\mu} = \tilde{\rho}(\chi_A)$. Using linearity of integral and $\tilde{\rho}$ we get $\tilde{\rho}(f) = \int f d\tilde{\mu}$. Finally using continuity of $\tilde{\rho}$, density of $\mathcal{A}lg$ in $C(\beta X)$ and bounded convergence theorem we get

$$\tilde{\rho}(f) = \int f d\tilde{\mu}, \forall f \in C(\beta X).$$

Let $\mu = \tilde{\mu} \circ \phi^{-1}$ be the push-out of $\tilde{\mu}$ to a measure on $(X, \mathcal{B}_X)$ using the continuous map ϕ. It is easy to see that μ is regular. Let $f \in C(X; \mathbb{C})$. Then by the abstract change of variable theorem we have

$$\int f d\mu = \int f \circ \phi d\tilde{\mu} = \int C(\phi)(f) d\tilde{\mu}, \forall f \in C(X).$$

Therefore for all $f \in C(X)$ we have

$$\rho(f) = \rho \circ C(\phi)^{-1}(C(\phi)(f)) = \tilde{\rho}(C(\phi)(f)) = \int C(\phi)(f) d\tilde{\mu} = \int f d\mu.$$

Only thing remains to be shown is uniqueness of μ. Let ν be another regular Borel measure on X such that $\rho(f) = \int f d\nu, \forall f \in C(X)$. Thanks to the outer regularity to show $\mu(A) = \nu(A), \forall A \in \mathcal{B}_X$ it is enough to show that $\mu(U) = \nu(U)$ for all open U. Let us fix an open set U. By Urysohn lemma we know that given any compact set $K \subseteq U$, there exists a continuous function $f_K : X \to [0,1]$ such that $f_K(x) = 1, \forall x \in K$ and $supp(f_K) \subseteq U$. Therefore $\mu(K) \leqslant \rho(f_K) = \int f_K d\mu \leqslant \mu(U)$. By inner regularity, $\mu(U) = \sup\{\rho(f_K) : K \subseteq U, K \text{ is compact}\}$. Clearly the right hand side also describes $\nu(U)$. Hence we have $\mu(U) = \nu(U)$ for all open U. □

4.7 Markov-Kakutani fixed point theorem

Now we will discuss applications of Hahn-Banach separation theorems. We begin with a cute application of theorem (3.2.5) yielding a proof of Markov-Kakutani fixed point theorem for locally convex spaces. This argument is due to Dirk Werner.

Theorem 4.7.1 (Markov-Kakutani fixed point theorem)**.** *Let C be a compact convex set in a locally convex space E. A continuous map $T : C \to C$ is said to be affine if $T(\lambda x + (1-\lambda)y) = \lambda T(x) + (1-\lambda)T(y), \forall x, y \in C, \forall \lambda \in [0,1]$. Every commuting family $\{T_i\}_{i \in I}$ of continuous affine endomorphisms of C has a common fixed point.*

Lemma 4.7.2. *Let C be a compact convex set in a locally convex Hausdorff space E and let $T : C \to C$ be a continuous affine transformation. Then T has a fixed point.*

Proof. Let $\Delta = \{(x,x) : x \in C\}$ be the diagonal in C and $\Gamma = \{(x, Tx) : x \in C\}$. If T has no fixed point then $\Delta \cap \Gamma = \emptyset$. Both Δ and Γ are compact convex sets in $E \times E$. By the Hahn-Banach theorem (3.2.5) we get continuous linear functionals ϕ_1, ϕ_2 and $\alpha, \beta \in \mathbb{R}, \alpha < \beta$ such that

$$\Re(\phi_1(x) + \phi_2(x)) \leqslant \alpha < \beta \leqslant \Re(\phi_1(y) + \phi_2(Ty)).$$

Consequently $\Re(\phi_2(Tx) - \phi_2(x)) \geqslant \beta - \alpha > 0$. Iterating this inequality we get $\Re(\phi_2(T^n x) - \phi_2(x)) \geqslant n(\beta - \alpha) \to \infty$ for arbitrary $x \in C$. This makes the sequence $\{\Re\phi_2(T^n(x))\}_n$ unbounded contradicting the compactness of $\Re\phi_2(C)$. □

Proof of Markov-Kakutani fixed point theorem. Let C_i be the fixed points of T_i. Then $C_i \neq \emptyset, C_i$ is compact and convex. We need to show $\cap C_i \neq \emptyset$. But that will follow once we establish finite intersection property. Since $T_i T_j = T_j T_i, T_i(C_j) \subseteq C_j$. Hence $T_i|_{C_j}$ has a fixed point by lemma. In other words $C_i \cap C_j \neq \emptyset$. An obvious induction shows $\cap_{i \in F} C_i \neq \emptyset, \forall$ finite $F \subseteq I$. □

4.8 Bi-polar theorem

We have already seen the concepts of left and right polar. They allow us to describe closures of certain sets in locally convex spaces. There are various versions of this result. We will do the real version.

Definition 4.8.1. Let E be a real vector space. The real polar of a subset $A \subseteq E$ is defined as

$$A^r := \{\phi \in E^* | \sup_{x \in A} \phi(x) \leqslant 1\}.$$

The real prepolar of a set $A \subseteq E^*$ is defined as

$${}^r A := \{x \in E : \sup_{\phi \in A} \phi(x) \leqslant 1\}.$$

Remark 4.8.2. This concept is related but little different from the earlier notion of polar. The earlier notion is also referred as the absolute polar.

Theorem 4.8.3. *Let $A \subseteq E$ be a subset of a locally convex space E over $\mathbb{R}$. Then ${}^r(A^r) = \overline{\mathrm{Co}}(\{0\} \cup A)$. In other words closure of the convex hull of $\{0\} \cup A$ is the real pre-polar of the real polar of A. Instead of ${}^r(A^r)$ we will also use A^{rr}.*

Proof. Since A^{rr} is a closed convex set containing $0, A$ we have $A^{rr} \supseteq \overline{\mathrm{Co}}(\{0\} \cup A)$. Suppose the inclusion is proper and $x_0 \in A^{rr} \setminus \overline{\mathrm{Co}}(\{0\} \cup A)$. By corollary 3.2.3 we get a bounded linear functional $\phi \in E^*$ such that

$$0 \leqslant \sup\{\phi(y) | y \in \overline{\mathrm{Co}}(\{0\} \cup A)\} < \phi(x_0).$$

Let $\alpha > 0$ be an element of the open interval $(\sup\{\phi(y) | y \in \overline{Co}(\{0\} \cup A)\}, \phi(x_0))$. Then $\psi := \alpha^{-1}\phi \in A^r \subseteq E^*$ and $\psi(x_0) > 1$. Thus $x_0 \notin A^{rr}$, a contradiction. □

Exercise 4.8.4. Let E be an LCS and $A \subseteq E$. Then the closure of the convex hull of the balanced hull of A is given by $^\circ(A^\circ)$.

Here is another application of a similar argument.

Lemma 4.8.5 (Mazur). *Let $C \subseteq E$ be a convex subset of a locally convex space E. Then $\overline{C} = \overline{C}^w$, where the left hand side denotes the closure in the original topology of E and the right hand side denotes the closure of C in the weak topology of E.*

Proof. The weak topology is coarser than the original topology and consequently we have $\overline{C} \subseteq \overline{C}^w$. Suppose $\exists x_0 \in \overline{C}^w \setminus \overline{C}$. By corollary 3.2.3 we get a bounded linear functional $\phi \in E^*$ such that

$$\Re\phi(x_0) < \inf\{\Re\phi(y) | y \in \overline{C}\}.$$

Let $\alpha \in (\Re\phi(x_0), \inf\{\Re\phi(y) | y \in \overline{C}\})$. Consider $F =:= \{x \in E : \Re\phi(x) \geqslant \alpha\}$. Then $C \subseteq F$ and F is convex, weakly closed. Therefore $\overline{C}^w \subseteq F$. Since $x_0 \in \overline{C}^w$ we get $x_0 \in F$. But clearly $x_0 \notin F$. This contradiction completes the proof. □

4.9 Krein-Milman theorem

Definition 4.9.1. Let E be locally convex Hausdorff topological vector space and $K \subseteq E$ be a convex subset. A point $x \in K$ is called an extreme point if $x = ty + (1-t)z$, for some $t \in (0,1)$ implies $y = z = x$. The set of extreme points will be denoted by $\mathrm{Ext}(K)$.

Theorem 4.9.2 (Krein-Milman). *Let E be a Hausdorff, LCS and $K \subseteq E$ be a compact convex subset. Then K is the closed convex hull of the set of extreme points of K. In particular this means that the set of extreme points is nonempty.*

Definition 4.9.3. Let $K \subseteq E$ be a nonempty compact convex subset. A face of K is a nonempty closed, convex subset F of K such that $ty + (1-t)z \in F$ for some $t \in (0,1)$, $y, z \in K$ implies $y, z \in F$.

Proof. Let $\mathcal{F} := \{F \subseteq K | F \text{ is a face of } K\}$. Consider the partial order $F_1 \succeq F_2$ iff $F_1 \subseteq F_2$. By Zorn's lemma choose a maximal element $F \in \mathcal{F}$. We will showthat F is a singleton $\{x\}$ for some x. Then it will follow that $\mathrm{Ext}(K) \neq \emptyset$. Suppose $x \neq y$ are two elements of F. Since E is Hausdorff, E^* separates E. There exists $\phi \in E^*$ such that $\Re\phi(x) < \Re\phi(y)$. Let $\alpha = \sup\{\Re\phi(u) : u \in F\}$ and $F' = \{z \in F | \Re\phi(z) = \alpha\}$. Since F is compact the supremum is attained and F' is nonempty. Therefore F' is a face of F and is properly contained in F because $x \in F \setminus F'$. This contradicts maximality of F. Therefore F must be a singleton set.

Let $L := \overline{\mathrm{Co}}\mathrm{Ext}(K)$. Since K is closed and convex we have $L \subseteq K$. Assume $x_0 \in K \setminus L$. Then by corollary 3.2.3 there exists $\phi \in E^*$ such that $\Re\phi(x_0) > \sup_{x \in L} \Re\phi(x)$. Let $\alpha = \sup_{x \in K} \Re\phi(x)$ and $F = \{x \in K | \Re\phi(x) = \alpha\}$. Then F is a face of K. Let $z \in \mathrm{Ext}(F) \subseteq \mathrm{Ext}(K)$. Note that $F \cap L = \emptyset$ because $\alpha \geqslant \phi(x_0)$. So, $z \notin L$. This contradicts $\mathrm{Ext}(K) \subseteq L$. □

We will close our discussion on Hahn-Banach and its applications by proving Banach-Stone theorem as an application of Krein-Milman theorem.

Theorem 4.9.4 (Banach-Stone)**.** *Let K, L be compact Hausdorff spaces. Then $C(K)$ is isometrically isomorphic with $C(L)$ iff K is homeomorphic with L.*

Remark 4.9.5. There is a bit of ambiguity in our notation for $C(K), C(L)$ etc. We have not specified the field of scalars. We will prove it for real scalars. That means for us $C(K), C(L)$ denotes $C(K, \mathbb{R}), C(L, \mathbb{R})$. We do this because we have not talked about complex measures. However we will write our argument in a manner so that it works for the complex scalar case as well.

Proof. Given $x \in K$ we denote by δ_x the Dirac delta mass at x. The extreme points of the unit ball of $C(K)$ are precisely measures of the form $\lambda(x)\delta_x$ where $\lambda(x) \in \mathbb{K}, |\lambda(x)| = 1$. Let $T : C(K) \to C(L)$ be an isometric isomorphism. Then $T^*(B_{C(L)^*}) = B_{C(K)^*}$. □

Chapter 5

Baire Category Theorem and its Consequences

For a play we need a stage and the actors. We have got our stage namely topological vector spaces. In fact we have various classes of them like, LCTVS, Banach spaces, Hilbert spaces etc. We have also got some idea about continuous linear maps. These are the actors of the play. Now let us ask what was the first result we learnt in our linear algebra course and can we extend it to this framework. We could have developed the theory in the generality of Frechet spaces but simplicity demands instead we do it in the framework of Banach spaces.

5.1 Baire Category Theorem

Theorem 5.1.1 (Baire Category Theorem)**.** *Let X be a complete metric space. If* U_n *is a sequence of open dense sets in* X *then* $\cap U_n$ *is also dense in* X*.*

Proof. Let d be a distance defining the topology of X. Let B be an open ball and we want to show that $B \cap U_n \neq \phi$. Clearly it suffices to show that for any closed ball $\bar{B} \cap U_n \neq \phi$. Replacing X by $\bar{B}$ it suffices to show that $\cap U_n \neq \phi$. We shall define a sequence x_n and positive real numbers r_n such that (i) $B'(x_n, r_n) \subseteq U_n \cap B(x_{n-1}, r_{n-1})$ and (ii) $r_n < 1/n$. Here $B'(u, r)$ denotes the closed ball with center u and radius r. Start with $x_1 \in U_1$ and $r_1 < 1$ such that $B'(x_1, r_1) \subseteq U_1$. After defining $X_1, \cdots, x_{n-1}$ choose

$x_n \in U_n \cap B(x_{n-1}, r_{n-1})$ and $r_n < 1/n$ such that (ii) holds. One can do this because U_n is dense and $U_n \cap B(x_{n-1}, r_{n-1})$ is open. Clearly $d(x_n, x_{n+p}) < r_n < 1/n$ for each $n \geqslant 1$ and p. Hence x_n is a Cauchy sequence and by hypothesis it converges to some $x \in E$. Since $x_{n+p} \in B'(x_n, r_n)$ for all $p > 1$, $x \in B'(x_n, r_n) \subseteq U_n$ for each n. Therefore $x \in \cap U_n$. □

Corollary 5.1.2. Let X be a complete metric space and C_n a sequence of closed sets such that $X = \cup C_n$. Then at least one of them has nonempty interior.

Proof. On the contrary suppose every C_n has empty interior. Let $U_n = X \setminus C_n$, then U_n's are dense open subsets of X and by Baire's theorem $\cap U_n$ is dense. On the other hand

$$\cap U_n = \cap(X \setminus C_n) = X \setminus (\cup C_n) = X \setminus X = \phi$$

a contradiction. □

5.2 The uniform boundedness principle and an application

Theorem 5.2.1 (Uniform Boundedness Principle)**.** *Let* $\{T_\alpha : E \to F\}_{\alpha \in A}$ *be a family of continuous linear maps such that for each* $x \in E$ *there exists* M_x *such that* $\sup_\alpha \|T_\alpha(x)\| \leqslant M_x\|x\|$*, then there exists* M *such that* $\sup_\alpha \|T_\alpha\| \leqslant M$.

Proof. Let $C_n = \{x \in E : \forall \alpha, \quad \|T_\alpha(x)\| \leqslant n\|x\|\}$. Then clearly each C_n is closed and they cover E. Therefore at least one of them say C_k contains a ball of radius r around x_0 for some r and x_0. Hence $\|T_\alpha(x\| \leqslant k\|x\|$ whenever $\|x - x_0\| < r$ and consequently for x with $\|x - x_0\| \leqslant r$ using $\|x\| \leqslant \|x_0\| + r$ we get

$$\|T_\alpha(x - x_0)\| \leqslant \|T_\alpha(x)\| + \|T_\alpha(x_0)\| \leqslant k\|x\| + k\|x_0\| \leqslant k(2\|x_0\| + r).$$

Therefore $\sup_\alpha \|T_\alpha\| \leqslant \frac{k(2\|x_0\|+r)}{r}$. □

Corollary 5.2.2. Let E be a Banach space. Let X be a weakly bounded subset of E. That means for all $\phi \in E^*$, $\phi(X)$ is a bounded subset of $\mathbb{K}$. Then X is a norm bounded subset of E.

Proof. Let $j : E \to E^{**}$ be the canonical embedding. Then by hypothesis

$$\forall \phi \in E^*, \exists M_\phi \text{ such tat } \sup_{x \in X} \|j(x)(\phi)\| < M_\phi.$$

By the uniform boundedness principle there exists M such that

$$\sup_{x \in X} \|x\| = \sup_{x \in X} \|j(x)\| < M.$$

□

5.3 A typical application

Let $1 < p < \infty$ and $\{\alpha_n\}$ be a sequence of scalars such that $\sum \alpha_n \beta_n$ converges for all $\{\beta_n\} \in \ell_p$. Then $\{\alpha_n\} \in \ell_q$. To see this consider the linear functional $T_N \in \ell_p^*$ given by $T_N(\{\beta_n\}) = \sum_{n=1}^{N} \alpha_n \beta_n$. From convergence of $\sum \alpha_n \beta_n$ we conclude that the hypothesis of UBP is met. Therefore UBP gives us M such that $M > \sup_N \|T_N\| = \sup_N \sqrt[q]{\sum_{n=1}^{N} |\alpha_n|^q}$. Therefore $\sum_{n=1}^{\infty} |\alpha_n|^q \leqslant M < \infty$.

5.4 Quotient spaces

Now that we have some idea about bounded linear maps on normed linear spaces we can ask how about extending some of the results of linear algebra to normed linear spaces. The first theorem we learnt was the first isomorphism theorem. Recall that first isomorphism theorem says if T is a linear map from a linear space E onto another linear space F then T induces an isomorphism $q_T : E/\ker T \to F$. Now if we want to extend this to normed linear spaces first thing we need is the notion of quotients.

Definition/Proposition 5.4.1. Let E be a normed linear space and $F \subseteq E$ a closed subspace. Then $\|[x]\| := \inf\{\|x + y\| : y \in F\}$ defines a norm on the vector space E/F.

Proof. Let $x_1, x_2 \in E$. Then $\forall y_1, y_2 \in F$ we have

$$\|x_1 + y_1 + x_2 + y_2\| \leqslant \|x_1 + y_1\| + \|x_2 + y_2\|.$$

Taking infimum over both sides as y_1, y_2 varies over F we get $\|[x_1+x_2]\| \leqslant \|[x_1]\|+\|[x_2]\|$. Similarly we get $\|[\lambda x]\| = |\lambda|\|[x]\|$. Finally note that $\|[x]\| = 0$ iff $x = \lim y_n$ for some sequence $\{y_n\} \subseteq F$. Since F is closed, this happens iff $x \in F$. In other words $[x] = 0 \in E/F$. □

Lemma 5.4.2. *Let E be a normed linear space. Then E is complete iff convergence of $\sum \|x_n\|$ implies convergence of $\sum x_n$.*

Proof. Only if part is easy and we only show the if part. Let $\{x_n\}$ be a Cauchy sequence in E. Then we can extract a subsequence $\{x_{n_k}\}$ such that $\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k}, \forall k$. Then the series $\sum \|x_{n_{k+1}} - x_{n_k}\|$ converges. By our hypothesis $\sum_{k=1}^{N}(x_{n_{k+1}} - x_{n_k})$ converges. That means $x_{n_k} - x_{n_1}$ converges. In other words the subsequence $\{x_{n_k}\}$ converges. Since the original sequence is Cauchy from the convergence of a subsequence we conclude convergence of the whole sequence. □

Proposition 5.4.3. Let E be a Banach space and $F \subseteq E$ is a closed subspace. Then E/F with the quotient norm is a Banach space.

Proof. Let $\sum \|[x_n]\| < \infty$ to show completeness of E/F it is enough to show convergence of $\sum[x_n]$. For each n obtain $y_n \in F$ such that $\|x_n + y_n\| \leqslant \|[x_n]\| + \frac{1}{2^n}$. Then $\sum \|x_n + y_n\| < \infty$ and using completeness of E we conclude convergence of $\sum(x_n+y_n)$ say to x_0. In other words $\|\sum_{n=1}^{N}(x_n+y_n) - x_0\| \to 0$. Since $\sum_{n=1}^{N} y_n \in F$ we have

$$\|\sum_{n=1}^{N}[x_n] - [x_0]\| \leqslant \|\sum_{n=1}^{N}(x_n + y_n) - x_0\| \to 0.$$

Thus we have established $\lim \sum_{n=1}^{N}[x_n] = [x_0]$. □

5.5 Open mapping theorem and its main corollary

Theorem 5.5.1 (Open Mapping Theorem). *Let $T : E \to F$ be a continuous surjection, then T is an open mapping.*

[Lecture Notes of P.S.Chakraborty]

Lemma 5.5.2. *Let $T : E \to F$ be a bounded operator from a Banach space E to another Banach space F. Let B_E and B_F be the unit balls of E and F respectively. Suppose that $T(B_E)$ closure contains rB_F for some $r > 0$, then $T(B_E)$ contains rB_F.*

Proof. Let $y \in rB_F$ and $\delta \in (0,1)$ such that $y' = \delta^{-1}y \in rB_F$. By the assumption, there exists $x_1 \in B_E$ such that $\|y' - T(x_1)\| < (1-\delta)r$. Since $\overline{T((1-\delta)B_E)}$ contains $(1-\delta)rB_F$, there exists $x_2 \in (1-\delta)B_E$ such that $\|y - T(x_1) - T(x_2)\| < r(1-\delta)^2$. Since $\overline{T((1-\delta)^2B_E)}$ contains $(1-\delta)^2rB_F$, there exists $x_3 \in (1-\delta)^2B_E$ such that $\|y - T(x_1) - T(x_2) - T(x_3)\| < r(1-\delta)^3$. Continuing this process we get a sequence $x_n \in (1-\delta)^{n-1}B_E$ such that

$$\|y - T(x_1) - T(x_2) - \cdots - T(x_n)\| < r(1-\delta)^n.$$

Since $\sum \|x_n\|$ converges and E is complete the series $\sum x_n$ converges to x' say. Since T is continuous $T(x') = y'$ and $\|x'\| < \sum (1-\delta)^{-1} = \delta^{-1}$. Put $x = \delta x'$, then clearly $x \in B_E$ and $T(x) = \delta y' = y$. □

Open Mapping Theorem. We have to show that the image of an open ball around zero under T contains an open ball around zero. Since T is surjective, $F = \cup T(nB_E)$. But by the corollary to the Baire theorem we get closure of $T(mB_E)$ contains an open ball $V = y + \epsilon B_F$. Put $r = \frac{\epsilon}{2m}$ and take $z \in rB_F$. Since $y, y + 2mz \in V$, there exists sequences $y_n, {y'}_n \in T(mB_E)$ such that $\lim y_n = y, \lim {y'}_n = y + 2mz$. Hence $z_n = y_n - {y'}_n \in T(2mB_E)$ converges to $2mz$, and thus $\frac{1}{2m}z_n \in T(B_E)$ converges to z. Thus we can apply the previous lemma and conclude the proof. □

Remark 5.5.3 (A typical application). Let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on a linear space E turning E into a Banach space. Suppose there exists $C > 0$ such that $\|x\|_1 \leqslant C\|x\|_2, \forall x \in E$. Then there exists C' such that $\|x\|_2 \leqslant C'\|x\|_1, \forall x \in E$. To see this just observe that the identity map from $(E, \|\cdot\|_2)$ to $(E, \|\cdot\|_1)$ is a bijective continuous surjection. By the open mapping theorem this mapping has a continuous or equivalently bounded inverse. We can take C' to be the norm of the inverse.

Theorem 5.5.4 (Closed Graph Theorem)**.** *Let E, F be Banach spaces and $T : E \to F$ a linear map such that the graph of T, $\Gamma = \{(x, T(x)) : x \in E\}$ is a closed subset of $E \times F$. Then T is continuous.*

Proof. The vector space $E \times F$ is a Banach space with the norm $\|(x, y)\| = \|x\|_E + \|y\|_F$. By hypothesis Γ is a closed subspace of a Banach space, hence Γ becomes a Banach space. Define $\pi_1 : \Gamma \to E$ as $\pi_1((x, T(x))) = x$ and $\pi_2 : E \times F \to F$, as $\pi_2((x, y)) = y$. By the open mapping theorem π_1^{-1} is a continuous linear map from X to Γ. But $T = \pi_2 \circ \pi_1^{-1}$, hence continuous. □

Proposition 5.5.5. Let $\|\cdot\|_{\mathcal{N}}$ be a norm on $C([0, 1])$ turning it into a Banach space. Also $\|f_n - f\|_{\mathcal{N}} \to 0$ implies $\lim f_n(x) = f(x), \forall x \in [0, 1]$. Then $\|\cdot\|_{\mathcal{N}}$ must be equivalent with the sup norm.

Proof. Because of remark (5.5.3) it is enough to show that the identity mapping from $(C([0, 1]), \|\cdot\|_{\sup})$ to $(C([0, 1]), \|\cdot\|_{\mathcal{N}})$ is continuous. We can appeal to closed graph theorem provided we show that the graph of identity mapping is closed. In other words if $\lim \|f_n - f\|_{\sup} = 0, \lim \|f_n - g\|_{\mathcal{N}} = 0$ then we must show $g = f$. But that follows from $g(x) = \lim f_n(x) = f(x)$. □

5.6 Practice problem set 3

1. Let E be a Banach space and F a finite dimensional subspace. Show that F is closed.

2. Let E be a finite dimensional Banach space. Can you give a dense proper subspace of E?

3. Let E be an infinite dimensional Banach space. Give a dense proper subspace of E.

4. Let E be a Banach space and F a closed subspace. We say F is algebraically complemented if there is another closed subspace F' such that $F \oplus F' = E$. Suppose F is finite dimensional. Then show that F is algebraically complemented.

5. Let E be a Banach space and F a closed subspace. We say F is topologically complemented if it is algebraically complemented and the norm on E is equivalent to the norm on the ℓ_1-sum of F and F' where F, F' are endowed with norms obtained from E as its subspaces. Show that if a closed subspace is algebraically complemented then it is topologically complemented.

6. Let E be a Banach space and $\phi : E \to \mathbb{K}$ be an unbounded linear functional then show that $\ker \phi$ is dense in E.

7. Let E be a Banach space and $\phi : E \to \mathbb{K}$ be a linear map. If $\ker \phi$ is a dense proper subspace then show that ϕ must be unbounded.

8. Let E be a Banach space and $\phi : E \to \mathbb{K}$. Then $\ker \phi$ is closed iff ϕ is continuous.

9. Show that there is a bounded linear map $L : \ell_\infty \to \mathbb{R}$ such that

   (a) $\liminf \underline{\mathbf{x}} \leqslant L(\underline{\mathbf{x}}) \leqslant \limsup \underline{\mathbf{x}}$.

   (b) $L(\underline{\mathbf{x}}) = \lim x_n$ if $L(\underline{\mathbf{x}}) = \{x_n\}$ is a convergent sequence.

   (c) $L(\underline{\mathbf{x}}) = L(S(\underline{\mathbf{x}}))$ where $S : \ell_\infty \to \ell_\infty$ is the shift operator given by $S(\underline{\mathbf{x}})_n = (\underline{\mathbf{x}})_{n+1}$.

10. Let E, F be Banach spaces and $T_n \in \mathcal{L}(E; F)$ be such that for all $x \in E$, the sequence $\{T_n(x)\}$ is convergent. Then show that $\sup_n \|T_n\| < \infty$. Let $T(x) \coloneqq \lim T_n(x)$. Then show that $T \in \mathcal{L}(E; F)$. If $x_n \to x$, then show that $T_n(x_n) \to T(x)$.

11. Show that for each n, k there exists $C_{n,k} > 0$ such that for all polynomials P of degree less than or equal to n, in k variables with $\mathbb{K}$ coefficients we have
$$\sup_{x \in B(0;r) \subseteq \mathbb{R}^k} |P(x)| \leqslant C_{n,k} \int_{B(0;r)} \frac{|P(x)|}{\mathrm{Vol}(B(0;r))} dx.$$

12. Given any two isomorphic Banach spaces E, F define their Banach Mazur distance as
$$\delta_{BM}(E, F) \coloneqq \inf\{\|T\|.\|T^{-1}\| : T \in \mathcal{L}(E, F) \text{ is invertible with } T^{-1} \in \mathcal{L}(F, E)\}$$
Then show that $\delta_{BM}(E, F) \geqslant 1$ and $\delta_{BM}(E, F) = 1$ along with $\dim E < \infty$ implies E, F are linearly isometrically isomorphic.

13. Let $F \subseteq E$ be normed linear spaces with F closed and $q : E \to E/F$ be the quotient map. Note that q is a surjection of norm less than or equal to 1. Show that whenever we have a normed space G and $T \in \mathcal{L}(E; G)$ with $F \subseteq \ker T$ there exists unique $\widetilde{T} \in \mathcal{L}(E/F; G)$ such that $T = \widetilde{T} \circ q$.

14. Let $F \subseteq E$ be normed linear spaces with F closed. Suppose we have a normed linear space G and a surjective bounded linear map $q : E \to G$ with $F \subseteq \ker q$, $\|q\| \leqslant 1$ so that whenever we have a normed space H and $T \in \mathcal{L}(E; H)$ with $F \subseteq \ker T$ there exists unique $\widetilde{T} \in \mathcal{L}(G; H)$ such that $T = \widetilde{T} \circ q$, then G must be isomorphic with E/F.

15. Let E be a normed linear space and $F \subseteq E$ be a complete subspace. Show that E is complete provided so is E/F.

16. Let $F \subseteq E$ be a closed subspace of a Banach space. Show that $\Phi : (E/F)^* \to F^\perp := \{x^* \in E^* : \langle x^*, x\rangle = 0, \forall x \in F\}$ given by $\Phi(\phi)(x) = \phi([x])$ is a linear isometric one to one onto map.

17. Let $F \subseteq E$ be a closed subspace of a Banach space. Define $\Psi : F^* \to E^*/F^\perp$ as follows: given $\phi \in F^*$ by Hahn Banach obtain a norm preserving extension $\tilde{\phi}$. Define $\Psi(\phi) = [\tilde{\phi}]$. Show that Ψ is a linear isometric isomorphism.

18. Let $F \subseteq E$ be a closed subspace of a Banach space. If E is reflexive then show that so is E/F.

19. Let E be a reflexive Banach space. Show that for all $x^* \in E^*, \exists x \in E, \|x\| = 1, x^*(x) = \|x^*\|$.

20. Goal of this exercise is showing the collection of continuous nowhere differentiable functions is a dense G_δ subset of $C[0,1]$.

    (a) Let $\mathcal{F}_n = \{f \in C[0,1] : \exists x_f \in [0,1]$, such that $\forall y \in [0,1], |f(y) - f(x_f)| \leqslant n|y - x_f|, \}$. Then show that $\mathcal{F}_n$ is closed.

    (b) Let $f \in C[0,1]$ be differentiable at x. Then show that $f \in \cup_n \mathcal{F}_n$.

    (c) Finally show that $\mathcal{F}_n$ has empty interior.

    (d) Conclude that no where differentiable continuous functions form a dense G_δ subset of $C[0,1]$.

Chapter 6

Hilbert Spaces

We will introduce Hilbert spaces and develop them. Hilbert spaces are easy to classify but that is not the end of story so we indicate what makes Hilbert spaces interesting.

6.1 Inner product spaces

Let us quickly recall some notions you are already familiar with.

Definition 6.1.1. Let $\mathcal{H}$ be a vector space. A pre-inner product on $\mathcal{H}$ is a sesquilinear map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ such that

1. $\langle u, v \rangle = \overline{\langle v, u \rangle}, \forall u, v \in \mathcal{H}$.

2. $\langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle, \forall \alpha, \beta \in \mathbb{K}, \forall u, v \in \mathcal{H}$.

3. $\langle u, u \rangle \geqslant 0 \; \forall u \in \mathcal{H}$.

Definition 6.1.2. A <u>**Pre-Hilbert Space**</u> or a pre-inner product space is a pair consisting of vector space along with a pre-inner product.

Proposition 6.1.3 (Cauchy-Schwarz Inequality)**.** Let $\mathcal{H}$ be a vector space equipped with a pre-inner product, then

$$|<u,v>| \leqslant \sqrt{\langle u, u \rangle}\sqrt{\langle v, v \rangle}, \forall u, v \in \mathcal{H}.$$

Proof. Let $\langle u, v\rangle = re^{i\theta}, r \geqslant 0$. Note that if the scalar field is $\mathbb{R}$ then $\theta \in \{\pi, 0\}$. We will divide the proof in cases. The first one is$\langle u, u\rangle = \langle v, v\rangle = 0$.

$$\begin{aligned} 0 &\leqslant \langle u - e^{-i\theta}v, u - e^{-i\theta}v\rangle \\ &= \langle u, u\rangle + \langle v, v\rangle - e^{-i\theta}\langle u, v\rangle - e^{i\theta}\overline{\langle v, u\rangle} \\ &= -2r \leqslant 0. \end{aligned}$$

Thus we get $r = 0$ proving the inequality in this case. Next case is both $\langle u, u\rangle$ and $\langle v, v\rangle$ are not simultaneously zero. Without loss of generality we can assume that $\langle v, v\rangle \neq 0$. Let $t = -\frac{\overline{\langle u, v\rangle}}{\sqrt{\langle v, v\rangle}}$, then,

$$\begin{aligned} 0 &\leqslant \langle u + tv, u + tv\rangle \\ &= \langle u, u\rangle + |t|^2\langle v, v\rangle - \frac{2|\langle u, v\rangle|^2}{\langle v, v\rangle} \\ &= \langle u, u\rangle + \frac{|\langle u, v\rangle|^2}{\langle v, v\rangle} - \frac{2|\langle u, v\rangle|^2}{\langle v, v\rangle} \\ &= \langle u, u\rangle - \frac{|\langle u, v\rangle|^2}{\langle v, v\rangle}. \end{aligned}$$

Now transferring $\frac{|\langle u, v\rangle|^2}{\langle v, v\rangle}$ to the other side and multiplying both sides by $\langle v, v\rangle$ we get the result. □

Corollary 6.1.4. We have $\langle u, v\rangle = 0$ whenever $\langle v, v\rangle = 0$.

Corollary 6.1.5. $N = \{v \in \mathcal{H} : \langle v, v\rangle = 0\}$ is a subspace.

Proof. Clearly N is closed under scalar multiplication. Only thing we need to show that it is closed under addition. Let $u, v \in N$. Then by the C-S inequality we get $\langle u, v\rangle = 0$. Thus $\langle u + v, u + v\rangle = 0$. □

Corollary 6.1.6. $\sqrt{\langle u, u\rangle} = \sup_{v:\langle v, v\rangle = 1} |\langle u, v\rangle|$

Proof. If $\langle u, u\rangle = 0$ then both sides are zero. Otherwise by the C-S inequality left hand side is less than or equal to right hand side and taking $v = u/\sqrt{\langle u, u\rangle}$ we get the other inequality. □

Definition 6.1.7. Let $\mathcal{H}$ be a vector space. An inner product on $\mathcal{H}$ is a sesquilinear map $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ such that

1. $\langle\cdot,\cdot\rangle$ is a pre-inner product.

2. Positive definiteness: $\langle u,u\rangle = 0 \Longrightarrow u = 0$.

An inner product space $(\mathcal{H},\langle\cdot,\cdot\rangle)$ is a pair consisting of a vector space $\mathcal{H}$ along with an inner product on $\mathcal{H}$

Definition/Proposition 6.1.8. Let $(\mathcal{H},\langle\cdot,\cdot\rangle)$ be an inner product space, then the map $\|\cdot\| : \mathcal{H} \to \mathbb{R}_+$ given by

$$\|v\| = \begin{cases} \sqrt{\langle v,v\rangle}, v \neq 0 \\ 0, \text{ for } v = 0. \end{cases}$$

is a norm on $\mathcal{H}$. This norm is referred as the norm associated with the inner product $\langle\cdot,\cdot\rangle$.

Proof. Let $u, v \in \mathcal{H}$. Only thing we need to verify is $\|u+v\| \leqslant \|u\| + \|v\|$. That follows from,

$$\begin{aligned} \|u+v\|^2 &= \langle u+v, u+v\rangle = \|u\|^2 + \|v\|^2 + 2\Re(\langle u,v\rangle) \\ &\leqslant \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| = (\|u\| + \|v\|)^2 \end{aligned}$$

□

Definition 6.1.9. An inner product space $(\mathcal{H},\langle\cdot,\cdot\rangle)$ is called a **Hilbert space** if $\mathcal{H}$ is complete with respect to the norm associated with the inner product.

Definition 6.1.10. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. A linear map $U : \mathcal{H}_1 \to \mathcal{H}_2$ is called **unitary** if it is one-to-one, onto and preserves inner products that is, $\langle Ux, Uy\rangle = \langle x,y\rangle$, for all $x, y \in \mathcal{H}_1$. The Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ are called unitarily equivalent if there is a unitary U from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Proposition 6.1.11. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces with dense subspaces S_1, S_2 respectively. Let $U : S_1 \to S_2$ be a bijection such that $\langle Ux, Uy\rangle = \langle x,y\rangle$, for all $x, y \in S_1$, then U extends to a unitary map denoted by the same symbol U from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Proof. Observe that $\|U(x)\| = \|x\|$, for all $x \in \mathcal{S}_1$. Therefore U converts Cauchy sequences to Cauchy sequences. If x is an element in $\mathcal{H}_1$ there is a sequence $\{x_n\}$ of elements of S_1 converging to x. Now $\{U(x_n)\}$ is also Cauchy and therefore converges to some limit. Define Ux as this limit. Clearly this is well defined. By playing the same game with U^{-1} we conclude that the extended map is bijective as well. Continuity of the inner-product combined with the density of S_i's give $\langle Ux, Uy\rangle = \langle x, y\rangle$, for all $x, y \in \mathcal{H}_1$. □

Definition/Proposition 6.1.12. Let $(\mathcal{H}_{pre}, (\cdot,\cdot))$ be a pre-Hilbert space. Let $N = \{v \in \mathcal{H}_{pre} : (v,v) = 0\}$. Then $\langle u + N, v + N\rangle = (u,v)$ defines an inner product on $\mathcal{H}_{pre}/N$. Completion of $\mathcal{H}_{pre}/N$ with respect to the associated norm is called the Hilbert space associated with the pre-Hilbert space $\mathcal{H}_{pre}$.

Proof. By corollary (6.1.4) the sesquilinear form $\langle\cdot,\cdot\rangle$ is well defined. Only thing we need to verify is positive definiteness. Let $u \in \mathcal{H}_{pre}$ be such that $\langle u+N, u+N\rangle = (u,u) = 0$. Then $u \in N$ and consequently $u+N = N$. □

6.2 Key properties of Hilbert spaces

Now we will discuss key properties of Hilbert spaces.

Proposition 6.2.1. Let $\mathcal{H}$ be a Hilbert space and $C \subseteq \mathcal{H}$ be a closed convex set. Then for all $x \notin C$ there exists unique $\tilde{z} \in C$ such that $\|x - \tilde{z}\| = \inf\{\|x - z\| : z \in C\}$. Verbally this means C has a unique point closest to x.

Proof. Uniqueness: Let $z_1, z_2 \in C$ be equidistant from x. In other words $\|x - z_1\| = \|x - z_2\|$. Then by the parallelogram identity

$$\|(x - z_1) + (x - z_2)\|^2 + \|(x - z_1) - (x - z_2)\|^2 = 2(\|x - z_1\|^2 + \|x - z_2\|^2)$$

Therefore

$$\|x - \frac{z_1 + z_2}{2}\|^2 + \frac{1}{4}\|z_1 - z_2\|^2 = \|x - z_1\|^2 = \|x - z_2\|^2.$$

So, either $z_1 = z_2$ or else their midpoint $\frac{z_1+z_2}{2}$ is a point from C closer to x.

Existence: Let $c = \inf\{\|x - z\|^2 : z \in C\}$. Then there exists a sequence $\{z_n\} \subseteq C$ such that $c \leqslant \|x - z_n\|^2 \leqslant c + \frac{1}{n}$. Then using parallelogram identity we get

$$\begin{aligned}\|z_n - z_m\|^2 &= 2(\|x - z_n\|^2 + \|x - z_m\|^2) - 4\|x - \frac{z_1 + z_2}{2}\|^2 \\ &\leqslant 2(c + 1/n + c + 1/m) - 4c = 2(1/n + 1/m).\end{aligned}$$

Since $\mathcal{H}$ is complete and C is closed $\{z_n\}$ converges to some $\tilde{z} \in C$. Using continuity of corm we conclude

$$\|x - \tilde{z}\| = \lim \|x - z_n\| = c = \inf\{\|x - z\|^2 : z \in C\}.$$

□

Proposition 6.2.2. Let $\mathcal{H}_0 \subseteq \mathcal{H}$ be a closed subspace and $x \notin \mathcal{H}_0$. Let $\tilde{z}$ be the unique solution to the minimization problem $\min\{\|x - z\| : z \in \mathcal{H}_0\}$. Then $\langle x - \tilde{z}, z\rangle = 0, \forall z \in \mathcal{H}_0$.

Proof. We do it for complex scalars. The real case is easier. Let $\lambda \in \mathbb{C}$ and $z \in \mathcal{H}_0$. Then

$$\|x - \tilde{z}\|^2 \leqslant \|x - \tilde{z} - \lambda z\|^2$$

So, for all such λ and z

$$-2\Re\langle x - \tilde{z}, \lambda z\rangle + |\lambda|^2\|z\|^2 \geqslant 0.$$

Write $\lambda = |\lambda|e^{i\theta}$, fix θ, divide by $|\lambda|$ and let $|\lambda|$ go to zero to conclude

$$-2\Re\langle x - \tilde{z}, e^{i\theta}z\rangle \geqslant 0.$$

Since θ is arbitrary we must have $\langle x - \tilde{z}, z\rangle = 0$. □

Definition 6.2.3. Let $S \subseteq \mathcal{H}$ be a subset. Then $S^\perp := \{x \in \mathcal{H} : \langle x, y\rangle = 0, \forall y \in S\}$.

Proposition 6.2.4. Let $S \subseteq \mathcal{H}$ be a subset. Then the following holds.

1. $S^\perp$ is a closed subspace.
2. $S^{\perp\perp} := (S^\perp)^\perp$ is the closure of linear span of S.
3. $S \cap S^\perp \subseteq \{0\}$. If $0 \in S$ then $S \cap S^\perp = \{0\}$.

Proof. Obvious. □

Theorem 6.2.5 (Projection theorem)**.** *Let $\mathcal{H}_0 \subseteq \mathcal{H}$ be a closed subspace. Then every $x \in \mathcal{H}$ can be written uniquely as $y + z$ where $y \in \mathcal{H}_0, z \in \mathcal{H}_0^\perp$. The mapping $P_{\mathcal{H}_0} : x \mapsto y$ is a bounded linear map from $\mathcal{H}$ to itself so that $P_{\mathcal{H}_0}^2 = P_{\mathcal{H}_0}$.*

Proof. Let $y = \arg\min\{\|x - u\| : u \in \mathcal{H}_0\}$ and $z = x - y \in \mathcal{H}_0^\perp$ by proposition (6.2.2). To see uniqueness of the decomposition note that if $x = y_1 + z_1 = y_2 + z_2$ with $y_1, y_2 \in \mathcal{H}_0, z_1, z_2 \in \mathcal{H}_0^\perp$, then $y_1 - y_2 = z_2 - z_1 \in \mathcal{H}_0 \cap \mathcal{H}_0^\perp = \{0\}$. Clearly $P_{\mathcal{H}_0} : x \mapsto y$ is linear. To see it is bounded let us calculate $\|x\|^2$, keeping in mind $\langle y, z\rangle = 0$.

$$\|x\|^2 = \langle y + z, y + z\rangle = \langle y, y\rangle + \langle z, z\rangle = \|y\|^2 + \|z\|^2 \geqslant \|y\|^2 = \|P_{\mathcal{H}_0}(x)\|^2.$$

Therefore $P_{\mathcal{H}_0}$ is bounded with norm bounded by 1. If $\mathcal{H}_0 \neq \{0\}$ then $\|P_{\mathcal{H}_0}\| = 1$. □

Theorem 6.2.6 (Riesz Representation Theorem)**.** *Let $\phi \in \mathcal{H}^*$, then there is unique $u_\phi \in \mathcal{H}$ so that $\phi(v) = \langle u_\phi, v\rangle$. Moreover $\|\phi\| = \|u_\phi\|$. The mapping $\phi \mapsto u_\phi$ gives a conjugate linear isometry from $\mathcal{H}^*$ to $\mathcal{H}$.*

Proof. Let $\mathcal{H}_0 = \ker\phi$. Note that $\phi = 0$ if and only if $\ker\phi = \mathcal{H}$. So, if $\phi = 0$ we can take $u_\phi = 0$. Let us now consider the case $\phi \neq 0$. Then $\mathcal{H}_0$ is a proper subspace. So there exists $v \in \mathcal{H}_0^\perp$ satisfying $\phi(v) = 1$. By the first isomorphism theorem of linear algebra $\mathcal{H}_0^\perp = \mathbb{C}v$. Let $u_\phi = \frac{v}{\|v\|^2}$, then

$$\langle u_\phi, w\rangle = \begin{cases} 0, \text{if} & w \in \mathcal{H}_0 \\ 1, \text{if} & w \in \mathcal{H}_0^\perp. \end{cases}$$

Thus $\phi(w) = \langle u_\phi, w\rangle, \forall w$. An application of Cauchy-Schwarz inequality yields $\|\phi\| = \|u_\phi\|$. □

Definition 6.2.7. Let $\mathcal{H}$ be a Hilbert space.

1. Orthogonal set: A subset $S \subseteq \mathcal{H}$ is said to be orthogonal if every element of S is nonzero and $v, w \in S, v \neq w$ implies $\langle v, w\rangle = 0$.

2. Orthonormal set: A subset $S \subseteq \mathcal{H}$ is said to be orthonormal if it is orthogonal and every element of S has norm one.

3. Orthonormal basis: A maximal with respect to inclusion orthonormal set is called an orthonormal basis to be abbreviated as O.N.B. It exists by a simple application of Zorn's lemma.

4. An orthonormal set S is said to be complete if $\mathcal{H} = \overline{\mathrm{Span}\, S}$.

Definition 6.2.8. Let X be a set and $f : X \to \mathbb{R}_{\geqslant 0}$ be a function. Let $\mathcal{F} := \{F \subseteq X : F \text{ is a finite set}\}$. This is directed by inclusion. The limit of the net $\{s_F := \sum_{x \in F} f(x)\}_{F \in \mathcal{F}}$ if exists is denoted by $\sum_{x \in X} f(x)$.

Theorem 6.2.9 (Bessel's inequality)**.** *Let $\mathfrak{B}$ be an orthonormal set. Then for all $v \in \mathcal{H}$ we have $\sum_{u \in \mathfrak{B}} |\langle u, v\rangle|^2 \leqslant \|v\|^2$.*

Proof. Let $F \subseteq \mathfrak{B}$ be a finite subset. Then $\{\langle u, v\rangle u : u \in F\} \cup \{v - \sum_{u \in F} \langle u, v\rangle u\}$ is an orthogonal set and by exercise (6.2.10) we have

$$\sum_{u \in F} \|\langle u, v\rangle u\|^2 + \|v - \sum_{u \in F} \langle u, v\rangle u\|^2 = \|v\|^2.$$

Therefore $\sum_{u \in F} \|\langle u, v\rangle u\|^2 \leqslant \|v\|^2$. The net $F \mapsto \sum_{u \in F} \|\langle u, v\rangle u\|^2$ is a montone net bounded by $\|v\|^2$. Hence it converges to $\sum_{u \in \mathfrak{B}} |\langle u, v\rangle|^2 \leqslant \|v\|^2$. □

Exercise 6.2.10. Let S be a finite orthogonal set. Then $\| \sum_{u \in S} u\|^2 = \sum_{u \in S} \|u\|^2$.

Proposition 6.2.11. Every orthonormal set can be extended to an orthonormal basis.

Proof. Let $\mathfrak{B}$ be an orthonormal set. Consider the partially ordered set $\mathcal{P} := \{\mathfrak{B}' : \mathfrak{B}' \supset \mathfrak{B}, \mathfrak{B}' \text{ is an O.N.B}\}$ ordered by inclusion. Clearly every chain in this partially ordered set has an upper bound it has a maximal element $\mathfrak{B}'$. This gives an orthonormal basis containing $\mathfrak{B}$. □

Lemma 6.2.12. *Let S be an orthonormal set and $x \in \mathcal{H}$, then the orthogonal projection of x on span of S is given by $\sum_{v \in S} \langle v, x\rangle v$.*

Proof. Note that $\langle x - \sum_{v \in S} \langle v, x\rangle v, w\rangle = 0, \forall w \in S$. Therefore

$$\begin{aligned} \|x - \sum_{v \in S} \lambda_v v\|^2 &= \|x - \sum_{v \in S} \langle v, x\rangle v + \sum_{v \in S} (\lambda_v + \langle v, x\rangle) v\|^2 \\ &= \|x - \sum_{v \in S} \langle v, x\rangle v\|^2 + \sum_{v \in S} |(\lambda_v + \langle v, x\rangle)|^2 \quad \text{[By pythagoras} \\ &\geqslant \|x - \sum_{v \in S} \langle v, x\rangle v\|^2 \end{aligned} \tag{6.1}$$

Thus $\sum_{v\in S}\langle v,x\rangle v = \arg\min\{\|x-u\| : u \in \mathrm{Span}\, S\}$. □

Proposition 6.2.13. Let $S \subseteq \mathcal{H}$ be an orthonormal set then the following are equivalent.

1. S is an orthonormal basis.
2. S is complete.
3. For all $x \in \mathcal{H}$, $\|x\|^2 = \sum_{v\in S} |\langle v,x\rangle|^2$

Proof. $(1) \implies (2)$: Let $\mathcal{H}_0$ be the closed linear span of S. If $\mathcal{H}_0 \subsetneq \mathcal{H}$, then choose $v \in \mathcal{H} \setminus \mathcal{H}_0$. The vector $w := v - P_{\mathcal{H}_0} v$ must be non-zero because otherwise $v = P_{\mathcal{H}_0} v \in \mathcal{H}_0$. Since $w \in \mathcal{H}_0^{\perp}$, $S \cup \{\frac{w}{\|w\|}\}$ is an orthonormal basis properly containing S. This contradicts maximality of S!

$(2) \implies (3)$: Let $x \in \mathcal{H}$. Then for any finite set $F \subseteq S$, $(x - \sum_{v\in F}\langle v,x\rangle v) \perp v, \forall v \in F$. Therefore by pythagoras' theorem

$$\|x\|^2 = \sum_{v\in F} |\langle v,x\rangle|^2 + \|x - \sum_{v\in F}\langle v,x\rangle v\|^2 \tag{6.2}$$

Using completeness of S, for each $\epsilon > 0$ we get $v_1, \ldots, v_{n(\epsilon)} \in S$ and scalars $\lambda_1, \ldots, \lambda_{n(\epsilon)}$ so that $\|x - \sum_{j=1}^{n(\epsilon)} \lambda_j v_j\| < \epsilon$. If we call the finite set $\{v_1, \ldots, v_{n(\epsilon)}\}, F_\epsilon$ then by (6.1)

$$\|x - \sum_{v\in F_\epsilon}\langle v,x\rangle v\|^2 \leqslant \|x - \sum_{j=1}^{n(\epsilon)} \lambda_j v_j\|^2 < \epsilon^2 \tag{6.3}$$

Therefore the net $F \mapsto x - \sum_{v\in F_\epsilon}\langle v,x\rangle v$ defined on the directed set of finite subsets of S converges to 0. In other words the second term in (6.2) converges to 0. This proves $\|x\|^2 = \lim_F \sum_{v\in F} |\langle v,x\rangle|^2$.

$(3) \implies (1)$: If possible let $x \in \mathcal{H} \setminus S$ be such that $\{x\} \cup S$ be orthonormal. Then $\langle v,x\rangle = 0, \forall v \in S$. Therefore $\|x\|^2 = \sum_{v\in S} |\langle v,x\rangle|^2 = 0$, a contradiction to orthonormality of $\{x\} \cup S$. □

Corollary 6.2.14 (Abstract Fourier Expansion)**.** Let S be an orthonormal basis. Then for all $x \in \mathcal{H}$ we have $x = \sum_{v\in S}\langle v,x\rangle v$.

Proof. Since $\|x\|^2 = \lim_F \sum_{v\in F} |\langle v,x\rangle|^2$, from (6.2) we have $\lim_F \|x - \sum_{v\in F}\langle v,x\rangle v\| = 0$ or equivalently $x = \lim_F \sum_{v\in F}\langle v,x\rangle v =: \sum_{v\in S}\langle v,x\rangle v$. □

[Lecture Notes of P.S.Chakraborty]

Proposition 6.2.15. Any two o.n.b have same cardinality.

Proof. Let $\mathcal{H}$ be a Hilbert space with two orthonormal basis A, B. We will prove the proposition in the infinite dimensional case only. Fix a countable dense subset $\mathbb{K}'$ of $\mathbb{K}$. Let,

$$\mathcal{H}_A = \{v \in \mathcal{H} | \{a \in A : \langle v, a \rangle \neq 0\} \text{ is finite and } \langle v, a \rangle \in \mathbb{K}', \forall a \in A\}$$

Then $\mathcal{H}_A$ is dense in $\mathcal{H}$ and is in bijection with $\cup_{n=1}^{\infty} A^n \times \mathbb{K}'^n$ which is in bijection with A. Define $f : B \to \mathcal{H}_A$, such that $\|b - f(b)\| < 1/8$, for all $b \in B$. Orthonormality of B implies $\|b - b'\| > 1$ whenever we have two distinct elements of B. Thus given any two distinct elements $b, b' \in B$ we have $\|f(b) - f(b')\| > 1/2$. That is to say that f is one to one. This shows that the cardinality of A is greater than or equal to that of B. By symmetry we get the other inequality and conclude both A and B have the same cardinality. □

Proposition 6.2.16. Let $\mathcal{H}$ be a separable Hilbert space. Then any o.n.b is countable.

Proof. Fix a countable dense set S. Let A be an o.n.b. Define a function $f : A \to S$ such that $\|f(\alpha) - \alpha\| < 1/2$. Then f is one to one because, given any two distinct α, α' of A, we have

$$\|f(\alpha) - f(\alpha')\| \geqslant \|\alpha - \alpha'\| - \|f(\alpha) - \alpha\| - \|f(\alpha') - \alpha'\| > 1 - 1/2 - 1/2 = 0.$$

This shows that A is countable. □

6.3 Applications of Riesz representation theorem

Proposition 6.3.1. Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ be a sesquilinear form. If there exists a positive constant C such that

$$|T(u, v)| \leqslant C\|u\|\|v\|, \forall u, v \in \mathcal{H}.$$

Then there is a unique bounded linear map $\widetilde{T} \in B(\mathcal{H})$ such that $\|\widetilde{T}\| \leqslant C$ and

$$T(u, v) = \langle \widetilde{T}(u), v \rangle, \forall u, v \in \mathcal{H}.$$

Proof. Consider the linear map $\phi_u : \mathcal{H} \to \mathbb{K}, v \mapsto T(u, v)$. Then by the Riesz representation theorem there exists $\widetilde{T}(u)$ such that $\|\widetilde{T}(u)\| = \|\phi_u\|$ and $\langle \widetilde{T}(u), v\rangle = \phi_u(v) = T(u, v)$ for all $v \in \mathcal{H}$. Of course we need to verify that the map $u \mapsto \widetilde{T}(u)$ is linear. Note that

$$\begin{aligned}
\langle \widetilde{T}(\alpha \cdot u + \beta \cdot v), w\rangle &= T(\alpha \cdot u + \beta \cdot v, w) \\
&= \overline{\alpha} T(u, w) + \overline{\beta} T(v, w) \\
&= \langle \alpha \cdot \widetilde{T}(u), w\rangle + \langle \beta \cdot \widetilde{T}(v), w\rangle. \\
&= \langle \alpha \cdot \widetilde{T}(u) + \beta \cdot \widetilde{T}(v), w\rangle.
\end{aligned}$$

Thus we have

$$\langle \widetilde{T}(\alpha \cdot u + \beta \cdot v) - \alpha \cdot \widetilde{T}(u) - \beta \cdot \widetilde{T}(v), w\rangle = 0, \quad \forall w \in \mathcal{H}$$

In particular taking $w = \widetilde{T}(\alpha \cdot u + \beta \cdot v) - \alpha \cdot \widetilde{T}(u) - \beta \cdot \widetilde{T}(v)$ we get

$$\widetilde{T}(\alpha \cdot u + \beta \cdot v) - \alpha \cdot \widetilde{T}(u) - \beta \cdot \widetilde{T}(v) = 0.$$

That is to say that $\widetilde{T}$ is a linear map. To see that it is bounded note that

$$\|\widetilde{T}(u)\| = \|\phi_u\| = \sup_{v : \|v\| = 1} |T(u, v)| \leqslant C\|u\|.$$

Uniqueness of $\widetilde{T}$ is obvious because if there were two such maps $\widetilde{T}_1$ and $\widetilde{T}_2$, then

$$\langle \widetilde{T}_1(u) - \widetilde{T}_2(u), v\rangle = T(u, v) - T(u, v) = 0, \forall v \in \mathcal{H}.$$

Again taking $v = \widetilde{T}_1(u) - \widetilde{T}_2(u)$ we see that $\widetilde{T}_1(u) = \widetilde{T}_2(u)$. □

Remark 6.3.2. Similarly one can show that if we have Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, and a sesquilinear map $B : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{K}$ such that

$$|B(u, v)| \leqslant C\|u\|\|v\|, \forall u \in \mathcal{H}_1, \forall v \in \mathcal{H}_2$$

where C is a positive constant then there exists a bounded linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ of norm less than or equal to C and

$$B(u, v) = \langle T(u), v\rangle, \forall u \in \mathcal{H}_1, \forall v \in \mathcal{H}_2.$$

Corollary 6.3.3. Let $T \in B(\mathcal{H})$. Then there is a unique linear map denoted by T^* such that

$$\langle T^*(u), v\rangle = \langle u, T(v)\rangle, \quad \forall u, v \in \mathcal{H}. \tag{6.4}$$

Moreover $\|T^*\| = \|T\|$ and $T^{**} = T$.

Proof. Note that $|\langle u, T(v)\rangle| \leqslant \|T\|\|u\|\|v\|$. So, we can apply the previous proposition to the sesquilinear form $(u, v) \mapsto \langle u, T(v)\rangle$ to obtain a linear map T^* such that (6.4) holds. To see the statement about the norms,

$$\begin{aligned}\|T^*\| &= \sup_{u:\|u\|=1} \|T^*(u)\| \\ &= \sup_{u:\|u\|=1} \sup_{v:\|v\|=1} |\langle T^*(u), v\rangle| \\ &= \sup_{u:\|u\|=1} \sup_{v:\|v\|=1} |\langle u, T(v)\rangle| \\ &= \sup_{v:\|v\|=1} \|T(v)\|, \quad \text{by corollary (6.1.6)} \\ &= \|T\|.\end{aligned}$$

□

Corollary 6.3.4. In the set up of proposition (6) there exists a unique bounded linear map $T' \in B(\mathcal{H})$ such that $\|T'\| \leqslant C$ and

$$T(u, v) = \langle u, T'(v)\rangle, \forall u, v \in \mathcal{H}.$$

Proof. Take $T' = \widetilde{T}^*$. □

Definition 6.3.5. A bounded linear operator $T \in B(\mathcal{H})$ is called self-adjoint if $T = T^*$

Exercise 6.3.6. A bounded linear map U on a Hilbert space is a unitary iff $U^*U = UU^* = I$, where I stands for the identity operator.

6.4 Practice problems set 4

1. Let E be a real Banach space and $U : E \to E$ a bijective map such that

$$\|Ux - Uy\| = \|x - y\|, \forall x, y \in E.$$

Such maps will be referred as bijective isometry. Fix $x_1, x_2 \in E$. For any bijective isometry ϕ define

$$\mathrm{def}(\phi) = \|\phi(\frac{x_1 + x_2}{2}) - \frac{\phi(x_1) + \phi(x_2)}{2}\|$$

(a) Show that $\mathrm{def}(\phi) \leqslant \frac{\|x_1 - x_2\|}{2}$.

(b) Let $\rho_U : E \to E$ be $\rho_U(z) = U(x_1) + U(x_2) - z$ and $U' = U^{-1} \circ \rho_U \circ U$. Then $\mathrm{def}(U') = 2\mathrm{def}(U)$.

(c) Conclude $\mathrm{def}(U) = 0$ and U is affine.

2. Let E be a Banach space. Let X be a weakly bounded subset of E. That means for all $\phi \in E^*$, $\phi(X)$ is a bounded subset of $\mathbb{K}$. Then X is a norm bounded subset of E.

3. Let $\mathcal{H}$ be a Hilbert space and $u, v \in \mathcal{H}$. Let $\mathcal{H}_u, \mathcal{H}_v$ be the spans of u, v respectively. Suppose

$$\|v - u\| = \inf\{\|v - w\| : w \in \mathcal{H}_u\} = \inf\{\|u - w\| : w \in \mathcal{H}_v\},$$

then show that $v = u$.

4. We know that every closed convex subset in a Hilbert space has a unique element of maximum norm. However this exercise shows there may not be any element of maximum norm. Let $\{\phi_n : n \in \mathbb{N}\}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. Let

$$\mathcal{C} := \left\{ x \in \mathcal{H} | \sum \left(1 + \frac{1}{n}\right)^2 |\langle x, \phi_n \rangle|^2 \leqslant 1 \right\}.$$

Show that $\mathcal{C}$ is a closed, bounded and convex set, but it contains no vector of maximal norm. (Hint: Define $T(x) = \sum(1 + \frac{1}{n})\langle \phi_n, x \rangle \phi_n$.

Then $T \in \mathcal{B}(\mathcal{H})$ and $\mathcal{C} = \{x \in \mathcal{H} : \|T(x)\| \leqslant 1\}$. This shows $\mathcal{C}$ is closed and convex. Also for every

$$\|x\|^2 = \sum |\langle \phi_n, x\rangle|^2 < \sum \left(1 + \frac{1}{n}\right)^2 |\langle \phi_n, x\rangle|^2 \leqslant 1.$$

5. Let $S = \{\phi_n : n \in \mathbb{N}\} \subseteq L^2([0,1], d\lambda)$ be an orthonormal set. Show that the following are equivalent

   (a) S is an orthonormal basis.

   (b) For all $x \in [a, b]$, $\sum_{n=1}^{\infty} |\int_a^x \phi_n(t)dt|^2 = (x - a)$.

   (c) $\sum_{n=1}^{\infty} \int_a^b |\int_a^x \phi_n(t)dt|^2 dx = \frac{1}{2}(b-a)^2$.

6. Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ be a sesquilinear form. If there exists a positive constant C such that

$$|T(u,v)| \leqslant C\|u\|\|v\|, \forall u, v \in \mathcal{H}.$$

   Then there is a unique bounded linear map $\widetilde{T} \in B(\mathcal{H})$ such that $\|\widetilde{T}\| \leqslant C$ and

$$T(u,v) = \langle \widetilde{T}(u), v\rangle, \forall u, v \in \mathcal{H}.$$

7. If we have Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, and a sesquilinear map $B : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{K}$ such that

$$|B(u,v)| \leqslant C\|u\|\|v\|, \forall u \in \mathcal{H}_1, \forall v \in \mathcal{H}_2$$

   where C is a positive constant then there exists a bounded linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ of norm less than or equal to C and

$$B(u,v) = \langle T(u), v\rangle, \forall u \in \mathcal{H}_1, \forall v \in \mathcal{H}_2.$$

8. Let $x, y : [0,1] \to \mathbb{R}$ be C^1-functions such that $\|\frac{dx}{dt}\|_2^2 + \|\frac{dy}{dt}\|_2^2 = \ell^2$, then $|\int_0^1 y(t)\frac{dx}{dt}dt| \leqslant \frac{\ell^2}{4\pi}$.

9. The bilinear form T is called coercive if $\exists a > 0$ such that $T(u,u) \geqslant a\|u\|^2, \forall u \in \mathcal{H}$. By exercise (6) we know that there exists $\widetilde{T} \in B(\mathcal{H})$ such that $T(u,v) = \langle \widetilde{T}(u), v\rangle$. If T is given to be coercive.

(i) Show that $\widetilde{T}$ is one to one.

(ii) Let $\mathfrak{Ran}$ be the range of $\widetilde{T}$. Consider $S : \mathfrak{Ran} \to \mathcal{H}$ given by $S(u) = v$ where $u = \widetilde{T}(v)$. Show that S is bounded. and using this show that $\mathfrak{Ran}$ is closed.

(iii) Show that $\widetilde{T}$ is onto i.e., $\mathfrak{Ran} = \mathcal{H}$.

(iv) Conclude given $\phi \in \mathcal{H}$ there exists unique $u \in \mathcal{H}$ such that $T(u, v) = \langle \phi, v \rangle, \forall v \in \mathcal{H}$.

10. Let $(\Omega, \mathfrak{S}, \mu)$ be a probability space and $\mathfrak{S}' \subseteq \mathfrak{S}$ a sub-σ-algebra. Let f be a nonnegative measurable L_1 function. Let $L_2(\mathfrak{S}')$ be the space of square integrable $\mathfrak{S}'$ measurable functions. Then $L_2(\mathfrak{S}') \subseteq L_2(\mathfrak{S})$ is a closed subspace. Let P be the corresponding projection. Show that

(a) If $0 \leqslant f \leqslant C$ then $\exists N \in \mathfrak{S}', \mu(N) = 0$ and a $\mathfrak{S}'$ measurable g such that on $N^c, 0 \leqslant g \leqslant C$ and $g = Pf$ a.e. Such a g will be called a version of Pf.

(b)

$$\int_A f d\mu = \int_A P f d\mu, \forall A \in \mathfrak{S}'.$$

(c) Let $f_n = f \wedge n$, then $\exists N \in \mathfrak{S}', \mu(N) = 0$ such that outside N, each Pf_n has a version g_n such that $0 \leqslant g_n \leqslant n$ and $g_n \leqslant g_{n+1}, \forall n \geqslant 1$. Let $g = \lim g_n$. Show that

$$\int_A f d\mu = \int_A g d\mu, \forall A \in \mathfrak{S}'. \tag{6.5}$$

Such a g is called the conditional expectation of f given $\mathfrak{S}'$ and is denoted by $\mathbb{E}(f|\mathfrak{S}')$. This is an $\mathfrak{S}'$ measurable integrable function unique upto a μ null set.

Chapter 7

Spectral theory

In this chapter we explore the structure of linear operators on Hilbert spaces. We will begin with compact operators and prove spectral theorem. Then we move on to bounded operators and prove spectral theorem. Finally if time permits we will move to spectral theorem for unbounded self-adjoint operators.

7.1 Compact operators

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. We are interested in exploring the structure of $\mathcal{L}(\mathcal{H}, \mathcal{K})$, the collection of bounded linear maps from $\mathcal{H}$ to $\mathcal{K}$. To begin our exploration we begin by asking examples of bounded linear maps. Those examples may lead to further curiosities and the ball will get rolling.

Let $\mathcal{H}$ be a Hilbert space and $u \in \mathcal{H}$. The vector u has two kinds of life. On the one hand we can think of it as an element of $\mathcal{H}$. If we wish to emphasize this roll we use the notation $|u\rangle$ instead of u and we read $|u\rangle$ as a ket-vector. On the other we can also think of u as a linear functional on $\mathcal{H}$ and by the Riesz representation theorem every linear functional arises in this manner. When we wish to emphasize this roll we use the notation $\langle u|$ instead of u and call it a bra-vector. Given a bra-vector $\langle u|$ from $\mathcal{H}$ and a ket-vector $|v\rangle$ from $\mathcal{H}$, the action of the linear functional associated with bra-vector $\langle u|$ on the ket-vector $|v\rangle$ is not denoted by $\langle u|(|v\rangle)$. It is instead denoted by $\langle u, v\rangle$. This notation is due to Paul Dirac. Given a pair of vectors $u \in \mathcal{H}, v \in \mathcal{K}$ where $\mathcal{H}, \mathcal{K}$ are Hilbert spaces, $|v\rangle\langle u| : \mathcal{H} \to \mathcal{K}$

stands for the linear map $w \mapsto \langle u, w\rangle v$. In particular $P_u := |u\rangle\langle u|$ is the orthogonal projection onto the span of u.

Let us check whether $|v\rangle\langle u| : \mathcal{H} \to \mathcal{K}$ is bounded or not? Let $w \in \mathcal{H}$. Then

$$\|(|v\rangle\langle u|)(|w\rangle)\| = \|\langle u, w\rangle v\| = |\langle u, w\rangle|\|v\| \leqslant \|u\|\|v\|\|w\|.$$

Taking supremum as w varies with $\|w\| = 1$ we get $|v\rangle\langle u|$ is bounded with $\||v\rangle\langle u|\| \leqslant \|u\|\|v\|$. An operator of the form $|\langle u, w\rangle|\|v\|$ is called a rank one operator. A finite linear combination of rank one operators is called a finite rank operator. Given a bounded linear map how can we tell whether it is a finite rank operator or not? That's answered in the following exercise.

Exercise 7.1.1. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. A bounded linear map $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is a finite rank operator iff $T(\mathcal{H}) \subseteq \mathcal{K}$ is a finite dimensional subspace.

Remark 7.1.2. We have finite rank operators and a mechanism to recognise them. This is an instance of a recognition principle, albeit a rather easy one.

Now that we have a supply of bounded linear maps namely finite rank ones we can ask several questions.

1. Is the collection of finite rank operators closed?
2. If not then we can get more bounded operators by forming the closure of the set of finite rank operators.
3. Can there be a recognotion principle for the closure?

Exercise 7.1.3. Show that the set of finite rank operators is not closed.

To describe the closure we need the following concept.

Definition 7.1.4. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. A linear map $T : \mathcal{H} \to \mathcal{K}$ is said to be compact if $\overline{T(B_{\mathcal{H}})}$ is compact, where $B_{\mathcal{H}}$ is the unit ball of $\mathcal{H}$. The set of compact operators from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $\mathcal{B}_0(\mathcal{H}, \mathcal{K})$ or $\mathcal{L}_0(\mathcal{H}, \mathcal{K})$.

Remark 7.1.5. It is immediate that the concept of a compact operator can be defined for maps between Banach spaces as well. We have defined it for Hilbert spaces because we are not going to study the Banach space case.

Example 7.1.6. If T is a finite rank operator then by the Heine-Borel theorem image of unit ball under T becomes relatively compact. Therefore T becomes compact.

Exercise 7.1.7. Show that a compact operator is bounded.

Proposition 7.1.8. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. Then $\mathcal{L}_0(\mathcal{H}, \mathcal{K})$ is a closed subspace of $\mathcal{L}(\mathcal{H}, \mathcal{K})$.

Proof. Let $\{T_n\} \subseteq \mathcal{L}_0(\mathcal{H}, \mathcal{K})$ be a sequence converging to $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. We have to show that T is compact. Let $\epsilon > 0$. Enough to find an ϵ-net for $T(B_{\mathcal{H}})$. Get an N so that $\|T - T_N\| < \epsilon/3$. Since T_N is compact there exists a finite set $F \subseteq B_{\mathcal{H}}$ so that

$$\forall x \in B_{\mathcal{H}}, \exists x_F \in F, \text{ such that } \|T_N(x) - T_N(x_F)\| < \epsilon/3.$$

Then for all $x \in B_{\mathcal{H}}$,

$$\begin{aligned} \|T(x) - T(x_F)\| &\leqslant \|T(x) - T_N(x)\| + \|T_N(x) - T_N(x_F)\| + \|T(x_F) - T_N(x_F)\| \\ &< \|T - T_N\|\|x\| + \frac{\epsilon}{3} + \|T - T_N\|\|x_F\| \\ &< \epsilon. \end{aligned}$$

In other words F is an ϵ-net for $T(B_{\mathcal{H}})$. □

Corollary 7.1.9. Let $T = \lim T_n$ be a limit of finite rank operators. Then T is compact.

Proof. Immediately follows once we note that finite rank operators are compact. □

Proposition 7.1.10. Let $T \in B(\mathcal{H})$ then T is compact if and only if T converts weakly convergent sequences to norm convergent sequences. That is

$$(\langle v, u_n \rangle \to \langle v, u \rangle, \forall v \in \mathcal{H}) \implies \|T(u_n) - T(u)\| \to 0.$$

Proof. Let $\{u_n\}_{\alpha \in A}$ be a weakly convergent sequence with u as its limit. The sequence $\{T(u_n)\}$ weakly converges to $T(u)$ because

$$\langle v, T(u_n) \rangle = \langle T^*(v), u_n \rangle \to \langle T^*(v), u \rangle = \langle v, T(u) \rangle$$

In order to utilize the hypothesis that T is a compact operator note that the set $\{u_n : n \in \mathbb{N}\}$ is weakly bounded. Hence by corollary (5.2.2) it is norm bounded. So there exists M such that $\sup\{\|u_n\| : n \in \mathbb{N}\} < M$. Since T is compact any subnet of $\{T(u_n)\}$ has a convergent subsequence and the limit must be $T(u)$, because $\{T(u_n)\}$ weakly converges to $T(u)$. Since the limit of the convergent subsequence of any given subsequence does not depend on the subsequence the original sequence must be convergent with the same limit, i.e., $\|T(u_n) - T(u)\| \to 0$.

Conversely, let $\{T(u_n)\}$ be a sequence in $T(B(0,1))$. By Banach-Alaoglu theorem we can conclude that $\{u_n\}$ has a weakly convergent subsequence$\{u_{n_k}\}$. Then the corresponding subsequence $\{T(u_{n_k})\}$ converges. This shows that $T(B(0,1))$ is relatively compact or equivalently has compact closure. □

Corollary 7.1.11. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $R \in \mathcal{L}(\mathcal{H}), S \in \mathcal{L}(\mathcal{K}), T \in \mathcal{L}_0(\mathcal{H}, \mathcal{K})$, then $TR \in \mathcal{L}_0(\mathcal{H}, \mathcal{K}), ST \in \mathcal{L}_0(\mathcal{H}, \mathcal{K})$.

Theorem 7.1.12. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space and $T \in B(\mathcal{H})$ be a nonzero self-adjoint compact operator, then*

$$\Lambda_+ = \sup\{\langle u, Tu\rangle : \|u\| = 1\} = \sup\{\langle u, Tu\rangle : \|u\| \leqslant 1\}$$

$$\Lambda_- = \inf\{\langle u, Tu\rangle : \|u\| = 1\} = \inf\{\langle u, Tu\rangle : \|u\| \leqslant 1\}$$

are attained. Let u_+, u_- be the vectors where Λ_+, Λ_- are attained, then at least one of the following holds,

$$Tu_\pm = \Lambda_\pm u_\pm.$$

Proof. Let $F(u) = \langle u, Tu\rangle$, then this is a real valued function because,

$$\overline{F(u)} = \langle Tu, u\rangle = \langle u, T^*u\rangle = \langle u, Tu\rangle = Fu.$$

Also for $\|u\| \leqslant 1$, $|F(u)| \leqslant \|u\|^2\|T\| \leqslant \|T\|$. Therefore $\Lambda_\pm$ makes sense. Let $\{u_n\}$ be a sequence such that $\|u_n\| \leqslant 1$ and $F(u_n) \to \Lambda_+$. Since a Hilbert space is reflexive by Banach-Alaoglu theorem its unit ball is weakly compact the sequence $\{u_n\}$ has a weakly convergent subsequence. Without loss of generality we can assume that $u_n \to u_+$, weakly. Then,

$$\begin{aligned} |F(u_n) - F(u_+)| &= |\langle u_n, Tu_n\rangle - \langle u_+, Tu_+\rangle| \\ &\leqslant |\langle u_n, Tu_n - Tu_+\rangle| + |\langle u_n - u_+, Tu_+\rangle| \\ &\leqslant \|Tu_n - Tu_+\| + |\langle u_n - u_+, Tu_+\rangle| \\ &\to 0. \end{aligned}$$

Since T is sompact the first term goes to zero and the second term goes to zero because $\{u_n\}$ weakly converges to u_+. Therefore $F(u_+) = \lim F(u_n) = \Lambda_+$. Let $\{e_n : n \geqslant 1\}$ be an infinite orthonormal set. Then $\{e_n\}$ weakly converges to zero, hence $\{T(e_n)\}$ converges to zero in norm. Therefore $\{F(e_n)\}$ converges to zero. Thus $\Lambda_+ \geqslant 0$. If $\|u_+\| < 1$ there exists $\epsilon > 0$ such that $\|(1+\epsilon)u_+\| = 1$, and $F((1+\epsilon)u_+) = (1+\epsilon)F(u_+) \geqslant F(u_+)$. Similarly we obtain u_- such that $F(u_-) = \Lambda_-$.

$\Lambda_\pm$ both can not be zero: Suppose that $\Lambda_+ = \Lambda_- = 0$. Then for any u of unit norm, $F(u) = 0$. Thus for any u, we get $\langle u, Tu\rangle = 0$. Then by polarization we get

$$2\langle v, Tu\rangle = \langle u+v, T(u+v)\rangle + i\langle u+iv, T(u+iv)\rangle = 0.$$

Therefore $T = 0$ a contradiction to $T \neq 0$! □

Without loss of generality we assume that $\Lambda_+ \neq 0$. Then $\langle u_+, Tu_+\rangle = \Lambda_+ > 0$. Therefore, $T(u_+) \neq 0$.
Claim: $v \in \mathcal{H}, \|v\| = 1, v \perp u_+ \implies v \perp Tu_+$

Proof of Claim: Let $v_\theta = (\mathrm{Cos}\theta)v + (\mathrm{Sin}\theta)u_+$, then $\|v_\theta\| \leqslant 1$ and

$$\begin{aligned} F(v_\theta) &= \mathrm{Cos}^2\theta.F(v) + \mathrm{Sin}^2\theta.F(u_+) + \mathrm{Cos}\theta\mathrm{Sin}\theta\langle v, Tu_+\rangle \\ &\quad +\mathrm{Sin}\theta\mathrm{Cos}\theta\langle u_+, Tv\rangle \\ &= \mathrm{Cos}^2\theta F(v) + \mathrm{Sin}^2\theta F(u_+) + \mathrm{Sin}2\theta\Re\langle v, Tu_+\rangle \end{aligned}$$

We know that the function $\theta \mapsto F(v_\theta)$ attains its maximum at $\theta = \pi/2$. Therefore

$$\frac{dF(v_\theta)}{d\theta}|_{\theta=\pi/2} = \Re\langle v, Tu_+\rangle = 0.$$

Instead of v if we put $\sqrt{-1}v$ we obtain $\Im\langle v, Tu_+\rangle = 0$. Therefore $\langle v, Tu_+\rangle = 0$. □

Thus, $Tu_+ \in u_+^{\perp\perp} = \mathbb{C}u_+$. Let $Tu_+ = \lambda u_+$, and

$$\Lambda_+ = F(u_+) = \langle u_+, Tu_+\rangle = \lambda\|u_+\|^2 = \lambda.$$

If $\Lambda_- \neq 0$ we similarly conclude that $Tu_- = \Lambda_- u_-$. □

Lemma 7.1.13. *Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then*

$$\|T\| = \sup\{\frac{|\langle u, Tu\rangle|}{\|u\|^2} : \|u\| \neq 0\}. \tag{7.1}$$

Proof. Let M be the right hand side of 7.1. By Cauchy-Schwarz inequality we see that $M \leqslant \|T\|$. Let $u, v \in \mathcal{H}$, then

$$\begin{aligned}\langle u+v, T(u+v)\rangle &= \langle u, Tu\rangle + \langle u, Tv\rangle + \langle v, Tu\rangle + \langle v, Tv\rangle \\ \langle u-v, T(u-v)\rangle &= \langle u, Tu\rangle - \langle u, Tv\rangle - \langle v, Tu\rangle + \langle v, Tv\rangle\end{aligned}$$

Subtracting and taking absolute values we get

$$2|\langle u, Tv\rangle + \langle v, Tu\rangle| = |\langle u+v, T(u+v)\rangle - \langle u-v, T(u-v)\rangle| \tag{7.2}$$

If T is the zero operator then clearly $\|T\| \leqslant M$. So, we can assume $T \neq 0$. Let u be an arbitrary unit vector such that $Tu \neq 0$. Let $v = \frac{Tu}{\|Tu\|}$. Then, $\langle u, Tv\rangle = \langle Tu, v\rangle = \|Tu\|$. Putting these in 7.2 we get

$$\begin{aligned}4\|Tu\| =& |\langle u+v, T(u+v)\rangle - \langle u-v, T(u-v)\rangle| \\ \leqslant& M(\|u+v\|^2 + \|u-v\|^2) \\ =& M2(\|u\|^2 + \|v\|^2) && \text{[by parallelogram identity} \\ =& 4M && \text{[since } \|u\| = \|v\| = 1.\end{aligned}$$

Therefore $\|T\| \leqslant M$, establishing the other inequality required to show (7.1). □

Theorem 7.1.14 (Spectral Theorem for Compact Self-adjoint Operator)**.** *Let $T \neq 0$ be a compact self-adjoint operator on $\mathcal{H}$. Then there exists a sequence $\{\lambda_n\}$*

of real numbers and a sequence of mutually orthogonal vectors $\{e_n\}$ *such that* $|\lambda_n| \to 0, \|e_n\| = 1 \forall n$ *and*

$$T = \sum \lambda_n |e_n\rangle\langle e_n|, \tag{7.3}$$

where the sum appearing in (7.3) is norm convergent. The expansion (7.3) is called a spectral resolution of T*.*

Proof. Let $T^{(0)} = T, \mathcal{H}^{(0)} = \mathcal{H}$. Now we will successively define

1. Hilbert spaces $\mathcal{H}^{(n)}$ for $n \geqslant 0$ such that $\mathcal{H}^{(n+1)} \subseteq \mathcal{H}^{(n)}$.
2. Compact self-adjoint operators $T^{(n)} : \mathcal{H}^{(n)} \to \mathcal{H}^{(n)}$.
3. Vectors $e_{n+1} \in \mathcal{H}^{(n)}$ orthogonal to $\mathcal{H}^{(n+1)}$ and scalars λ_{n+1} for $n \geqslant 0$.

This will be defined in a manner so that if $Q^{(n)}$ denotes the orthogonal projection onto $\mathcal{H}^{(n+1)}$ then

$$T^{(n+1)} = T^{(n)}Q^{(n)} = Q^{(n)}T^{(n)} \tag{7.4}$$

$$T^{(n)} = \lambda_{n+1}P_{e_{n+1}} + T^{(n+1)}, \text{ for } n \geqslant 0, \tag{7.5}$$

$$\|T^{(n+1)}\| \leqslant |\lambda_{n+1}|. \tag{7.6}$$

This is achieved through repeated applications of theorem (7.1.12). Assume that we have defined $(T^{(k)}, \mathcal{H}^{(k)})$ for $k \leqslant n$. If $T^{(n)} = 0$ then $T^{(n+1)} = 0, \lambda_{n+1} = 0, e_{n+1}$ an arbitrary unit vector in $\mathcal{H}^{(n)}$ and $\mathcal{H}^{(n+1)} = \mathcal{H}^{(n)} \cap \{e_{n+1}\}^{\perp}$,. Otherwise apply theorem (7.1.12) for the operator $T^{(n)}$.

$$(\lambda_{n+1}, e_{n+1}) = \begin{cases} (\Lambda_+(T^{(n)}), u_+(T^{(n)})), \text{ if } \Lambda_+(T^{(n)}) \geqslant -\Lambda_-(T^{(n)}) \\ (\Lambda_-(T^{(n)}), u_-(T^{(n)})) \text{ otherwise.} \end{cases}$$

Then $T^{(n)}e_{n+1} = \lambda_{n+1}e_{n+1}$ and consequently $\lambda_{n+1}P_{e_{n+1}} = T^{(n)}P_{e_{n+1}} = P_{e_{n+1}}T$. Let $Q^{(n)} = I_{\mathcal{H}^{(n)}} - P_{e_{n+1}}$ and $\mathcal{H}^{(n+1)}$ be the range of $Q^{(n)}$. If we take $T^{(n+1)} = T^{(n)}Q^{(n)}$ then all the conditions will be met. To see (7.6) observe that

$$\|T^{(n+1)}\| \leqslant \|T^{(n)}\| = |\lambda_{n+1}|, \text{ by lemma (7.1).}$$

Adding (7.5) for $0 \leqslant n \leqslant k$ we obtain,

$$T = \sum_{n=0}^{k} \lambda_{n+1} P_{e_{n+1}} + T^{(k+1)}$$

Since $\{e_n\}$ converges to zero weakly $|\lambda_n| = \|T(e_n)\|$ converges to zero. It follows from the inequality (7.6) that $\|T^{(n)}\|$ converges to zero. This proves (7.3). □

Definition 7.1.15. Let $T \in B(\mathcal{H})$, then λ is an eigenvalue of T with eigenvector $u \neq 0$ if $Tu = \lambda u$. The subspace $E_\lambda = \{u \in \mathcal{H} : Tu = \lambda u\}$ is called the eigenspace corresponding to the eigenvalue λ.

Corollary 7.1.16. Let $T \neq 0$ be a compact operator with a spectral resolution given by (7.3). Then $\lambda \neq 0$ is an eigenvalue iff λ equals one of the λ_n's. Also $E_\lambda = span\{e_n : \lambda_n = \lambda\}$.

Proof. Let A be the orthonormal set consisting of e_n's. Extend it to an orthonormal basis A'. Let $\lambda \neq 0$ be an eigenvalue with eigenvector u. Then by corollary (6.2.14) $u = \sum_n \langle e_n, u\rangle e_n + \sum_{\alpha \in A' \setminus A} \langle \alpha, u\rangle \alpha$. Therefore $Tu = \sum_n \lambda_n \langle e_n, u\rangle e_n$. On the other hand $\lambda u = \sum_n \lambda \langle e_n, u\rangle e_n + \sum_{\alpha \in A' \setminus A} \langle \alpha, u\rangle \alpha$. Using $Tu = \lambda u$ we obtain,

$$\langle \alpha, u\rangle = 0, \forall \alpha \in A' \setminus A \tag{7.7}$$

$$\lambda \langle e_n, u\rangle = \lambda_n \langle e_n, u\rangle, \forall n. \tag{7.8}$$

Equation (7.7) tells us u belongs to the closed linear span of e_n's. Hence there exists n such that $\langle e_n, u\rangle \neq 0$. Using equation (7.8) for that n we conclude $\lambda = \lambda_n$. □

Corollary 7.1.17. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $T \in \mathcal{B}_0(\mathcal{H}, \mathcal{K})$ be a compact operator. Then there exists a unique compact self-adjoint operator in $\mathcal{B}_0(\mathcal{H})$, denoted by $|T|$ and referred as the modulus of T so that $T^*T = |T|^2$.

Proof. Since T is compact, so is T^*T. Also $\langle u, T^*Tu\rangle = \|Tu\|^2 \geqslant 0, \forall u$. Therefore every eigenvalue of T^*T must be nonnegative. Let $T^*T = \sum \lambda_n P_n$ be the spectral resolution of T^*T, where λ_n's are the distinct eigenvalues. Then define $|T| = \sum \sqrt{\lambda_n} P_n$. Since $\lim \sqrt{\lambda_n} = 0$, this sum is norm convergent. Also $|T|^2 = T^*T$. Uniqueness follows because if S is any such operator then S must have the spectral resolution of $|T|$. □

Our next result is a structure theorem for arbitrary compact operators. This is done via polar decomposition of an operator tobe defined shortly.

Definition 7.1.18. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. A linear map $U : \mathcal{H} \to \mathcal{K}$ is said to be a partial isometry with initial space $\mathcal{H}' \subseteq \mathcal{H}$ and terminal/final space $\mathcal{K}' \subseteq \mathcal{K}$ if

1. $U(\mathcal{H}') = \mathcal{K}'$;
2. $U|_{\mathcal{H}'^{\perp}} = 0$ and
3. $\langle Uu, Uv\rangle = \langle u, v\rangle, \forall u, v \in \mathcal{H}'$.

Exercise 7.1.19. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $U : \mathcal{H} \to \mathcal{K}$ be a bounded linear map. Then show that the following are equivalent.

1. $UU^*U = U$.
2. $U^*UU^* = U^*$.
3. UU^* is an orthogonal projection.
4. U^*U is an orthogonal projection.

Also show that these are equivalent to saying that U is a partial isometry.

Theorem 7.1.20 (Polar decomposition)**.** *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $T : \mathcal{H} \to \mathcal{K}$ be a compact linear map. Then there is a partial isometry $U : \mathcal{H} \to \mathcal{K}$ with initial space $\overline{\mathfrak{Ran}(|T|)}$ and final space $\overline{\mathfrak{Ran}(T)}$ so that $T = U|T|$.*

Proof. It is enough to define an inner product preserving onto map from $\mathfrak{Ran}(|T|)$ to $\mathfrak{Ran}(T)$. Define $U : |T|u \mapsto Tu$. Then forall $u, v \in \mathcal{H}$ we have

$$\langle U(|T|u), U(|T|v)\rangle = \langle Tu, Tv\rangle = \langle u, T^*Tv\rangle = \langle u, |T|^2 v\rangle = \langle |T|u, |T|v\rangle.$$

This shows U is well defined and a partial isometry. The equation $T = U|T|$ is immediate. □

Remark 7.1.21. Later we will show that given any bounded operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ there exists unique selfadjoint operator whose square is T^*T. In other words $|T|$ makes sense in that generality. Given that fact the proof of polar decomposition works for bounded operators.

Corollary 7.1.22 (Singular Value Decomposition)**.** Let $T \neq 0$ be a compact operator. Then there exists countable orthonormal sets $\{e_n\}, \{f_n\}$ and a sequence of positive scalars $\{s_n(T)\}$, $s_n(T) \searrow 0$, such that

$$T = \sum_n s_n(T)|f_n\rangle\langle e_n| \tag{7.9}$$

where the sum is norm convergent. The scalar $s_n(T)$ is called the n-th singular value of T. A representation of the form 7.9 is called a singular value decomposition. Such a decomposition may not be unique, however $s_n(T)$'s are unique and $s_n(T)$ is called the n-th largest singular value of T.

Proof. Let $S = T^*T$. Then S is compact and nonzero because if $Tu \neq 0$ then $\langle u, Su\rangle = \|Tu\|^2 > 0$. Hence S is nonzero and eigenvalues of S must be nonnegative. Let $\{s_n(T)^2\}$ be the sequence of nonzero eigenvalues of S arranged in decreasing order repeated according to their multipliciities. Then there exists an orthonormal family $\{e_n\}$ so that $|T| = \sum s_n(T)|e_n\rangle\langle e_n|$. Let $f_n := U(e_n)$ where U is the partial isometry in the polar decomposition of T. Then

$$T = U(|T|) = \sum s_n(T)U(|e_n\rangle\langle e_n|) = \sum s_n(T)|f_n\rangle\langle e_n|,$$

where the second equality is justified by the facts that U is a bounded linear map and the sum is norm convergent. Uniqueness of $s_n(T)$'s are obvious because they are precisely the eigenvalues of $|T|$. □

Corollary 7.1.23. Let T be a compact operator on a Hilbert space $\mathcal{H}$. Thenfor all n we have $s_n(T) = s_n(T^*)$.

Proof. Let $T = \sum s_n(T)|f_n\rangle\langle e_n|$ be a singular value decomposition. Then $T^* = \sum s_n(T)|e_n\rangle\langle f_n|$. Therefore $TT^* = \sum s_n(T)^2|f_n\rangle\langle f_n|$. Consequently

$$s_n(T^*) = \sqrt{s_n(T)^2} = s_n(T).$$ □

Theorem 7.1.24 (Min-max principle)**.** *Let T be a self-adjoint compact operator with $\langle u, Tu\rangle \geqslant 0$ for all u. Such operators are called positive. Then $\lambda_n(T)$, the n-th largest eigenvalue of T satisfies*

$$\lambda_{n+1}(T) = \min_{\dim(S)=n} \max_{\substack{u \perp S \\ \|u\|=1}} \langle u, Tu\rangle.$$

Proof. We have to show inequalities in both directions. To show that the left hand side is less than or equal to the right hand side, given a subspace S of dimension n we have to show that there exists $u \in S^{\perp}$ of norm 1 with $\lambda_{n+1}(T) \leqslant \langle u, Tu \rangle$. For the other inequality suffices to show there exists a subspace S_0 of dimension n so that $\lambda_{n+1}(T) \geqslant \max_{\substack{u \perp S_0 \\ \|u\|=1}} \langle u, Tu \rangle$.

Let us begin with the second assertion. Let $T = \sum \lambda_n(T) |e_n\rangle\langle e_n|$ be the spectral resolution of T. Take $S_0 = span\{e_1, \ldots, e_n\}$. Then we have

$$\lambda_{n+1}(T) \geqslant \max_{\substack{u \perp S_0 \\ \|u\|=1}} \langle u, Tu \rangle.$$

For the first assertion let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of S. Suffices to find $a_1, \ldots, a_{n+1}$ with $\sum |a_j|^2 = 1$ so that $u := \sum a_i e_i$ is orthogonal to $v_j, \forall j$. Because such an u will satisfy

$$\langle u, Tu \rangle = \sum_{j=1}^{n+1} \lambda_j(T) |a_j|^2 \geqslant \lambda_{n+1}(T) \sum_{j=1}^{n+1} |a_j|^2 = \lambda_{n+1}(T).$$

Consider the $n \times (n+1)$ matrix whose (i, j)-th entry is $\langle v_i, e_j \rangle$. Rank of this matrix is at most n. Therefore there exists a vector $(a_1, \cdots, a_{n+1})^t$ in the null space of this matrix. In other words $\sum_{j=1}^{n+1} a_j \langle v_i, e_j \rangle = 0, \forall i$, or equivalently $\sum_{j=1}^{n+1} a_j e_j \perp v_i, \forall i$. □

Corollary 7.1.25. Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}_0(\mathcal{H})$ be a compact operator. For $n \in \mathbb{N}$, let $s_n(T)$ be the n-th largest singular value of T. Then for $n \in \mathbb{N}_0$,

$$s_{n+1}(T) = \min_{\dim(S)=n} \max_{\substack{u \perp S \\ \|u\|=1}} \|Tu\|.$$

Proof. Let $R = T^*T$. Then $s_n(T)^2 = \lambda_n(R)$ and by the Min-max principle,

$$\lambda_{n+1}(R) = \min_{\dim(S)=n} \max_{\substack{u \perp S \\ \|u\|=1}} \langle u, T^*Tu \rangle = \min_{\dim(S)=n} \max_{\substack{u \perp S \\ \|u\|=1}} \|Tu\|^2.$$

Using monotonicity of the square root function on the non-negative real axis we get the result. □

Corollary 7.1.26. Let $\mathcal{H}$ be a Hilbert space. The mapping

$$s_n : \mathcal{B}_0(\mathcal{H}) \ni T \mapsto s_n(T) \in \mathbb{R}$$

is continuous for all $n \in \mathbb{N}$. Here $s_n(T)$ denotes the n-th largest singular value of T and $\mathcal{B}_0(\mathcal{H})$ is endowed with the norm topology.

Proof. Let $\{T_k\} \subseteq \mathcal{B}_0(\mathcal{H})$ be a sequence converging to T. Then

$$\forall \epsilon > 0, \exists N \text{ such that } \|T_k - T\| < \epsilon, \forall k \geqslant N.$$

Therefore,

$$\|Tu\| - \epsilon < \|T_k u\| < \|Tu\| + \epsilon, \forall k \geqslant N, \forall u \text{ with } \|u\| = 1.$$

Hence by corollary 7.1.25 we get for all $n \in \mathbb{N}$, for all $k \geqslant N, |s_n(T) - s_n(T_k)| < \epsilon$. □

Corollary 7.1.27. Let T be a compact operator and R be a bounded operator. Then we have already seen that both TR and RT are compact. Also $s_n(TR) \leqslant \|R\| s_n(T), s_n(RT) \leqslant \|R\| s_n(T), \forall n \in \mathbb{N}$.

Proof. By the min-max principle we have

$$s_{n+1}(RT) = \min_{\dim(S)=n} \max_{\substack{u \perp S \\ \|u\|=1}} \|RTu\| \leqslant \|R\| \min_{\dim(S)=n} \max_{\substack{u \perp S \\ \|u\|=1}} \|Tu\| = \|R\| s_{n+1}(T).$$

Also,

$$s_n(TR) = s_n(R^*T^*) \leqslant \|R^*\| s_n(T^*) = \|R\| s_n(T).$$ □

Definition 7.1.28 (Trace class operators)**.** A compact operator T is said to be trace class if $\sum s_n(T) < \infty$. The collection of trace class operators on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{B}_1(\mathcal{H})$ or $\mathcal{L}_1(\mathcal{H})$.

7.2 Practice problems

1. Let $\mathcal{H}$ be a Hilbert space. Show that there are dense subspaces of codimension 1.

2. Let $(\Omega, \mathfrak{A}, P)$ be a probability space with $\mathfrak{A}$ countably generated. Then show that $L^2((\Omega, \mathfrak{A}, P))$ is separable.

3. Consider the normed linear spaces with respective norms given by,

$$\mathcal{H}_\pm = \{(c_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} n^{\pm 4}|c_n|^2 < \infty\},$$
$$\|(c_n)\|_\pm^2 = \sum |n|^{\pm 4}|c_n|^2.$$

Show that $(\mathcal{H}_\pm, \|\cdot\|_\pm)$ are Hilbert spaces. Let $\varphi \in \mathcal{H}_+^*$. Show that there exists $(d_n)_n = \psi \in \mathcal{H}_-$ such that $\varphi((c_n)_n) = \sum \overline{d}_n c_n$.

4. (*) Let Ω be an open connected subset of the complex plane. Let

$$B^2(\Omega) := \{f : \Omega \to \mathbb{C} | f \text{ is holomorphic} \int_\Omega |f(z)|^2 dz < \infty\}$$

(a) Show that $B^2(\Omega)$ is an inner product space with inner product

$$\langle f, g\rangle := \int_\Omega \overline{f(z)} g(z) dz.$$

Let $\|f\|_{B^2(\Omega)}$ be the associated norm.

(b) Fix $w \in \Omega$ and choose $R > 0$ such that closed ball of radius R centred at w is contained in Ω. Show that $|f(w)| \leqslant \frac{\|f\|_{B^2(\Omega)}}{R\sqrt{\pi}}$.

(c) Conclude that $B^2(\Omega)$ is complete.

(d) Show that for all $w \in \Omega$, $f \mapsto f(w)$ defines a bounded linear functional on $B^2(\Omega)$.

5. A reproducing kernel Hilbert space is a Hilbert space $\mathcal{H}$ of functions on a set, Ω, so that

(i) For any $f \neq 0$, there is $x \in \Omega$ with $f(x) \neq 0$.

(ii) For any $x \in \Omega$, there is $f \in \mathcal{H}$ with $f(x) \neq 0$.

(iii) For any $x \neq y \in \Omega$, there is $f \in \mathcal{H}$ with $f(x) \neq f(y)$.

(iv) For any $x \in \Omega$, there is C_x so that $|f(x)| \leqslant C_x \|f\|_{\mathcal{H}}, \forall f \in \mathcal{H}$.

Prove that

(a) for any $x \in \Omega$, there is $k_x \in \mathcal{H}$ so that $\forall f \in \mathcal{H}, f(x) = \langle k_x, f \rangle$;

(b) each $k_x \neq 0$ and $x \neq y \implies k_x \neq k_y$;

(c) finite linear combinations of $\{k_x\}_{x\in\Omega}$ is dense in $\mathcal{H}$;

(d) $K(y, x) = \overline{K(x, y)}$ where $K : \Omega \times \Omega \to \mathbb{C}$ is the function

$$K(x, y) = \langle k_y, k_x \rangle = k_x(y)$$

called the reproducing kernel of $\mathcal{H}$;

(e) for any $n \in \mathbb{N}, x_1, \cdots, x_n \in \Omega, \zeta_1, \cdots, \zeta_n \in \mathbb{C}$, we have

$$\sum_{i,j=1}^{n} \overline{\zeta}_i \zeta_j K(x_i, x_j) \geqslant 0.$$

Such a function is called a positive definite kernel. This is actually positive semidefinite though.

6. Let $K : \Omega \times \Omega \to \mathbb{C}$ be a positive definite kernel. For $x \in \Omega$, let $k_x : \Omega \to \mathbb{C}$ be the function $k_x(y) = K(x, y)$. Let $c_0(\Omega)$ be the functions on Ω of the form $\sum_{i=1}^{m} \zeta_i k_{x_i}$ for finitely many points $x_1, \cdots, x_m$. Consider the sesquilinear form $\langle \cdot, \cdot \rangle : c_0(\Omega) \times c_0(\Omega) \to \mathbb{C}$

$$\langle \sum_{i=1}^{n} \eta_i k_{x_i}, \sum_{i=1}^{n} \zeta_j k_{x_j} \rangle \coloneqq \sum_{i,j=1}^{n} \overline{\eta}_i \zeta_j K(x_i, x_j).$$

(a) Show that this is a well defined preinner product.

(b) Prove that for all $f \in c_0(\Omega)$, $\langle k_x, f \rangle = f(x), \forall x \in \Omega$.

(c) Let $\mathcal{H}$ be the Hilbert space associated with the preinner product space $c_0(\Omega)$. Show that $\mathcal{H}$ is a reproducing kernel Hilbert space.

7. Let $T \in \mathcal{B}(\mathcal{H})$. Show that the following are equivalent.

(a) $T = PT$ for some finite rank orthogonal projection P.

(b) $T = TQ$ for some finite rank orthogonal projection Q.

(c) $T = P'TQ'$ for some finite rank orthogonal projections P', Q'.

(d) T is a finite rank operator.

8. (*) Let T be a finite rank operator. Then show that both kernel and cokernel of $(I - T)$ have same dimension. Now show the same for compact operators.

7.3 Banach Algebras

Definition 7.3.1. A Banach algebra $\mathcal{A}$ is a Banach space along with an associative and distributive multiplication denoted $(a, b) \mapsto a.b$ such that $\|a.b\| \leqslant C\|a\|\|b\|$ for all $a, b \in \mathcal{A}$ for some positive C.

Remark 7.3.2. Let $\mathcal{A}$ be a Banach algebra. Then there exists an equivalent norm $\|.\|'$ on $\mathcal{A}$ such that for all $a, b \in \mathcal{A}$, $\|a.b\|' \leqslant \|a\|'\|b\|'$.

Proof. Suppose $\|a.b\| \leqslant C\|a\|\|b\| \forall a, b \in \mathcal{A}$
Case 1: $C < 1$, take $\|a\|' = \|a\|$
Case 2: $C > 1$, define $\|a\|' = C\|a\|$
In view of the above remark given any Banach algebra we will assume that the norm satisfies $\|a.b\| \leqslant \|a\|\|b\|$ for all $a, b \in \mathcal{A}$. □

Proposition 7.3.3. (1) Let $\mathcal{A}$ be a Banach algebra. Then $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ is a Banach algebra provided,

$$\begin{aligned} (x, \alpha).(y, \beta) &= (xy + \alpha y + \beta x, \alpha\beta) \\ \|(x, \alpha)\| &= \|x\| + |\alpha| \end{aligned}$$

(2) $x \mapsto (x, 0)$ gives an isometric embedding of $\mathcal{A}$ in $\widetilde{\mathcal{A}}$ as an ideal.
(3) $e = (0, 1)$ satisfies $(x, \alpha).e = e.(x, \alpha) = (x, \alpha)$ and $\|e\| = 1$.

Definition 7.3.4. A Banach algebra $\mathcal{A}$ with an element e such that $e.x = x.e = x \forall x \in \mathcal{A}$, $\|e\| = 1$ is called a unital Banach algebra.

Remark 7.3.5. The previous proposition says every Banach algebra can be isometrically embedded into a unital Banach algebra. Henceforth unless otherwise stated a Banach algebra means a unital Banach algebra.

Example 7.3.6. Let K be a compact Hausdorff space. $C(K)$ be the space of all continuous complex valued functions on K. For $f, g \in C(K)$, Define

$$\begin{aligned} (f+g)(p) &= f(p)+g(p) \\ (f.g)(p) &= f(p).g(p) \\ \|f\| &= sup_{p \in K}|f(p)| \end{aligned}$$

$C(K)$ is a commutative Banach algebra.

Example 7.3.7. Let E be a Banach space. Then $\mathcal{L}(E)$, the space of all bounded linear maps from E to itself is a Banach algebra under operator norm.

Example 7.3.8. Let K be a compact subset of $\mathbb{C}$ or $\mathbb{C}^n$ with nonempty interior. Then $\mathcal{A} = \{f \in C(K) : f|_{\text{interior of } K} \text{ is holomorphic}\}$ is a Banach algebra.

Proposition 7.3.9. Let G be a locally compact group. Let μ be a Haar measure on G. Recall that μ satisfies

$$\int f(gh)d\mu(h) = \int f(h)d\mu(h).$$

Then $\mathcal{A} = L_1(G, \mu)$ is a Banach algebra with multiplication defined by

$$(f_1 \star f_2) = \int f_1(g)f_2(g^{-1}h)d\mu(g).$$

Proof. (1) $f_1 \star f_2 \in L_1$:

$$\begin{aligned} \int |f_1 \star f_2(h)d\mu(h) &\leqslant \int\int |f_1(g)||f_2(g^{-1}h|d\mu(g)d\mu(h) \\ &= \int |f_1(g)|d\mu(g)\int |f_2(h)|d\mu(h) \\ &= \|f_1\|_1\|f_2\|_1 \end{aligned}$$

Therefore we have proved

$$\begin{aligned} f_1 \star f_2 &\in L_1(G) \text{and} \\ \|f_1 \star f_2\|_1 &\leqslant \|f_1\|_1\|f_2\|_1 \end{aligned}$$

(2) $(f_1 \star f_2) \star f_3 = f_1 \star (f_2 \star f_3)$:

$$\begin{aligned}(f_1 \star f_2) \star f_3(u) &= \int (f_1 \star f_2)(v) f_3(v^{-1} u dv \\ &= \int\int f_1(w) f_2(w^{-1}v) f_3(v^{-1}u) dw dv \\ &= \int\int f_1(w) f_2(v) f_3(v^{-1}w^{-1}u) dw dv \\ &= f_1 \star (f_2 \star f_3)(u)\end{aligned}$$

□

Example 7.3.10. Let $C^1[0,1]$ be the space of once continuously differentiable functions. Define $\|f\| = sup_{x\in[0,1]}|f(x)| + sup_{x\in[0,1]}|f'(x)|$. Then under pointwise multiplication $C^1[0,1]$ is a Banach algebra.

Theorem 7.3.11. Assume that $\mathcal{A}$ is a Banach space as well as a complex algebra with a unit element $e \neq 0$, in which multiplication is both left and right continuous. Then there is a norm on $\mathcal{A}$ which induces the same topology as the given one and makes $\mathcal{A}$ a Banach algebra.

Proof. Define $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{A})$ by, $\pi(x)(z) = xz$. Clearly $\pi(x)$ is linear. It is continuous because multiplication is given to be right continuous. $\|x\| = \|xe\| = \|\pi(x)(e)\| \leqslant \|\pi(x)\|\|e\|$, So π is one to one. We also have $\|\pi(x)\pi(y)\| \leqslant \|\pi(x)\|\|\pi(y)\|, \|\pi(e)\| = 1$. So $\pi(\mathcal{A})$ is a Banach algebra provided it is complete. For that it is enough to show that $\pi(\mathcal{A})$ is closed. For that suppose $\pi(x_n) \to T$ in $\mathcal{L}(\mathcal{A})$. Then $x_n = \pi(x_n)(e) \to T(e) = x$.

$$T(y) = lim \pi(x_n)(y) = lim x_n y = xy = \pi(x)(y)$$

by continuity of left multiplication. So $T = \pi(x)$. □

Definition 7.3.12. A linear map $\phi : \mathcal{A} \to \mathcal{B}$ is called a homomorphism if

$$\begin{aligned}\phi(xy) &= \phi(x)\phi(y), \quad \forall x, y \in \mathcal{A} \\ \|\phi(x)\| &\leqslant \|x\| \quad \forall x \in \mathcal{A}.\end{aligned}$$

A nonzero homomorphism into the complex numbers is called a complex homomorphism

Proposition 7.3.13. If ϕ is a complex homomorphism on a Banach algebra $\mathcal{A}$ then $\phi(e) = 1$ and $\phi(x) \neq 0$ for all invertible $x \in \mathcal{A}$.

Proof. For some $y \in \mathcal{A}$, $\phi(y) \neq 0$, $\phi(y) = \phi(y)\phi(e)$ gives $\phi(e) = 1$. $\phi(x)\phi(x^{-1}) = \phi(e) = 1$ gives $\phi(x) \neq 0$. □

7.4 Spectrum

Proposition 7.4.1. Let $x \in \mathcal{A}$ with $\|x\| < 1$ then $(I - x)$ is invertible.

Proof. The series $\sum_{n=0}^{\infty} x^n$ converges and is the inverse of $(I - x)$. □

Corollary 7.4.2. Let $G(\mathcal{A})$ be the set of invertible elements of a Banach algebra $\mathcal{A}$. Then $G(\mathcal{A})$ is an open subset of $\mathcal{A}$.

Proof. Let $x \in G(\mathcal{A})$. For $y \in \mathcal{A}$ with $\|y\| < \frac{1}{\|x\|^{-1}}$, $(x - y) = x^{-1}(I - x^{-1}y)$ is invertible by the previous proposition because $\|x^{-1}y\| \leqslant \|x^{-1}\|\|y\| < 1$. □

Definition 7.4.3. Let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$. Then the spectrum of x is defined as $\{\lambda \in \mathbb{C} : (\lambda - x) \text{ is not invertible}\}$. It is denoted by $\sigma_{\mathcal{A}}(x)$. We often drop the subscript $\mathcal{A}$. For a nonunital Banach algebra $\mathcal{A}$ the spectrum of an element x is defined as $\sigma_{\widetilde{\mathcal{A}}}(x)$ where $\widetilde{\mathcal{A}}$ is the unitization defined before.

Definition 7.4.4. The spectral radius $\rho(x)$ of $x \in \mathcal{A}$ is defined as

$$\rho(x) = sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

Definition 7.4.5 (The resolvent set)**.** The complement of spectrum of $x \in \mathcal{A}$ is called the resolvent of x and is also denoted by $\rho(x)$. We have also used same notation for spectral radius. Both notations are standard. You have to make out from the context.

Definition 7.4.6 (The resolvent function)**.** Let $x \in \mathcal{A}$. Then for $\lambda \in \rho(x)$, the function $\lambda \mapsto \mathcal{R}_\lambda(x) = (\lambda 1_{\mathcal{A}} - x)^{-1}$ is called the resolvent function.

Proposition 7.4.7. Let x be an element of a Banach algebra $\mathcal{A}$. Then $\sigma(x)$ is a nonempty closed and bounded subset of $\mathbb{C}$.

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Proof. $\sigma(x)$ is closed:
Enough to show that it's complement is open. Suppose λ is such that $(\lambda - x)$ is invertible. Then by the proof of the previous corollary the ball of radius $\frac{1}{\|\lambda - x\|^{-1}}$ around λ is contained in $\sigma(x)^c$. Hence $\sigma(x)^c$ is open.
$\sigma(x)$ is bounded:
If λ is such that $|\lambda| > \|x\|$ then $(\lambda - x) = \lambda(I - \frac{x}{\lambda})$ is invertible. Hence $\sigma(x)$ is contained in the ball of radius $\|x\|$.
$\sigma(x)$ is nonempty:
If possible let $\sigma(x)$ be empty. Then $f(\lambda) = (\lambda - x)^{-1}$ is a holomorphic function defined on the entire plane. For $\lambda > \|x\|$, we have

$$
\begin{aligned}
f(\lambda) &= \lambda^{-1}\left(I - \frac{x}{\lambda}\right)^{-1} \\
&= \lambda^{-1}\sum_{n=0}^{\infty} x^n \lambda^{-n}, \text{since} \|\frac{x}{\lambda}\| < 1 \\
\text{So}, \|f(\lambda)\| &\leqslant |\lambda|^{-1}\frac{|\lambda|}{|\lambda| - \|x\|} \\
&\leqslant \frac{1}{|\lambda| - \|x\|}
\end{aligned}
$$

Hence f is a bounded entire function. Therefore it must be constant. From the previous estimates we see that $\lim_{\lambda\to\infty} f(\lambda) = 0$. Hence f is the constant function 0. But 0 is not invertible so we get a contradiction. □

Theorem 7.4.8 (Gelfand-Mazur)**.** *Let $\mathcal{A}$ be a Banach algebra such that every nonzero element is invertible then $\mathcal{A} \cong \mathbb{C}$.*

Proof. Suppose $\lambda_1 \neq \lambda_2 \in \sigma(x)$, then $(x - \lambda_1) = 0 = (x - \lambda_2)$. Hence, $\sigma(x)$ consists of a single point say $\lambda(x)$, and $x = \lambda(x)I$. $x \mapsto \lambda(x)$ gives an isomorphism between $\mathcal{A}$ and $\mathbb{C}$. □

Lemma 7.4.9. Let R be a commutative ring over $\mathbb{C}$. Then ab is invertible iff a and b are invertible.

Proof. Suppose $c = (ab)^{-1} = (ba)^{-1}$. Then $a^{-1} = bc$ because, (i) $abc = 1$ (ii) $bca = abc = 1$, the first equality uses commutativity. □

Proposition 7.4.10. Let p be a polynomial. Then for any $x \in \mathcal{A}$, $\sigma(p(x)) = p(\sigma(x))$.

Proof. Let $\lambda \in \mathbb{C}$.

$$\begin{aligned} p(z) - \lambda &= c\prod(z - \lambda_i), \quad \text{for some } c \neq 0, \lambda_1, \cdots, \lambda_n \in \mathbb{C} \\ p(x) - \lambda &= c\prod(x - \lambda_i) \end{aligned}$$

$\sigma(p(x)) \subseteq p(\sigma(x))$:

$$\begin{aligned} \lambda \in \sigma(p(x)) &\implies \lambda_i \in \sigma(x) \text{ for some } i \\ &\implies \lambda \in p(\sigma(x)) \text{ since } \lambda = p(\lambda_i) \end{aligned}$$

$p(\sigma(x)) \subseteq \sigma(p(x))$:

$$\begin{aligned} \lambda \in p(\sigma(x)) &\implies \lambda = p(\mu) \text{ for some } \mu \in \sigma(x) \\ &\implies p(x) - \lambda = (x - \mu)q(x) \text{ for some polynomial } q \\ &\implies \lambda \in \sigma(p(x)) \text{ by the lemma above} \end{aligned}$$

□

Proposition 7.4.11. Let x be an element of the Banach algebra $\mathcal{A}$. Then the spectral radius satisfies $\rho(x) = \mathrm{Lim}\|x^n\|^{\frac{1}{n}} = \inf\|x^n\|^{1/n}$

Proof. By the previous lemma $\rho(x^n) = \rho(x)^n, \forall n \geqslant 1$, also $\rho(x) \leqslant \|x\|$. So, $\rho(x)^n = \rho(x^n) \leqslant \|x\|^n$ implying $\rho(x) \leqslant \inf\|x^n\|^{1/n} \leqslant \underline{\mathrm{Lim}}\|x^n\|^{\frac{1}{n}}$. To complete the proof it suffices to show $\overline{\mathrm{Lim}}\|x^n\|^{\frac{1}{n}} \leqslant \rho(x)$. Let ϕ be a continuous linear functional on $\mathcal{A}$. Then the resolvent

$$f(\lambda) = (\lambda - x)^{-1} = \lambda^{-1}(1 - \lambda^{-1}x)^{-1}$$

is holomorphic outside the disk of radius $\rho(x)$. So, $g(\lambda) = \lambda(1 - \lambda x)^{-1}$ is analytic inside the disk of radius $\frac{1}{\rho(x)}$. For $|\lambda| < \|x\|$ we have the power series expansion $g(\lambda) = \sum \lambda^{n+1}x^n$. The function $\lambda \mapsto (\phi \circ g)(\lambda)$ is holomorphic in the disk of radius $\frac{1}{\rho(x)}$. Hence it's Taylor series $\sum \phi(x^n)\lambda^{n+1}$ converges in this disk. Thus

$$|\phi(\lambda^n x^n)| \to 0 \qquad \text{if } |\lambda|\rho(x) < 1.$$

For each fixed ϕ and λ we have some constant $C(\lambda, \phi)$ such that

$$\mathrm{Sup}_n|\phi(\lambda^n x^n)| < C(\lambda, \phi).$$

For each $|\lambda| < \frac{1}{\rho(x)}$ consider the family of linear functionals on $\mathcal{A}^*$ given by $T_n : \phi \mapsto \phi(\lambda^n x^n)$. We know

$$\mathrm{Sup}_n|T_n(\phi)| < C(\lambda, \phi).$$

By the uniform boundedness principle we get

$$\sup_n \|T_n\| < C(\lambda) \text{ for some constant } C(\lambda).$$

Clearly $\|T_n\| = \|\lambda^n x^n\|$, so

$$\begin{aligned} & \|x^n\| < C(\lambda)|\lambda|^{-n} \text{ for } |\lambda| < \frac{1}{\rho(x)} \\ \implies & \|x^n\|^{1/n} < C(\lambda)^{1/n}|\lambda|^{-1} \text{ for } |\lambda| < \frac{1}{\rho(x)} \\ \implies & \overline{\mathrm{Lim}}\|x^n\|^{1/n} < |\lambda|^{-1} \text{ for } |\lambda| < \frac{1}{\rho(x)} \\ \implies & \overline{\mathrm{Lim}}\|x^n\|^{1/n} \leqslant \rho(x) \end{aligned}$$

□

7.5 Abelian Banach Algebras

In this section unless otherwise stated we are dealing with a not necessarilly unital commutative Banach algebra $\mathcal{A}$.

Definition 7.5.1. An ideal $\mathfrak{m}$ of $\mathcal{A}$ is called regular the quotient ring $\mathcal{A}/\mathfrak{m}$ is unital. In other words if there exists $e \in \mathcal{A}$ such that $\forall x \in \mathcal{A}, \quad ex - x \in \mathfrak{m}$.

Proposition 7.5.2. Let $\mathfrak{m}$ be a proper regular ideal of $\mathcal{A}$. If e is an identity modulo $\mathfrak{m}$, then we have

$$\inf\{\|e - x\| : x \in \mathfrak{m} \geqslant 1.$$

Proof. Suppose $\|e - x\| < 1$ for some $x \in \mathfrak{m}$. Then the power series $y = \sum_{n=1}^{\infty} (e - x)^n$ converges. Since $(e - x)y = \sum_{n \geqslant 2} (e - x)^n$, we have

$$\begin{aligned} y &= (e - x) + (e - x)y \\ &= ey - xy + e - x. \end{aligned}$$

Hence $e = y - ey + xy + x \in \mathfrak{m}$. For any $a \in \mathcal{A}$, $a = ea + (a - ea) \in \mathfrak{m}$. Thus $\mathfrak{m} = \mathcal{A}$, a contradiction! □

Corollary 7.5.3. The closure of any regular proper ideal of an abelian Banach algebra $\mathcal{A}$ is proper and regular. In particular any maximal regular ideal is closed.

Proposition 7.5.4. Any proper regular ideal is contained in a maximal regular ideal.

Proof. Let e be an identity modulo $\mathfrak{m}$. Then any ideal containing $\mathfrak{m}$ is regular. Now apply Zorn's lemma to ideals containing $\mathfrak{m}$ and not containing e. □

Proposition 7.5.5. Let $\mathfrak{m}$ be a closed ideal of a possibly noncommutative Banach algebra $\mathcal{A}$. The quotient algebra $\mathcal{A}/\mathfrak{m}$ is a Banach algebra.

Proof. Let $\pi : \mathcal{A} \to \mathcal{A}/\mathfrak{m}$ be the quotient map. From the definition of the quotient norm it follows that $\|\pi(x)\| = \inf\{\|x + m\| : m \in \mathfrak{m}\}$. Given $\epsilon > 0$ get m, n from $\mathfrak{m}$ such that $\|x + m\| \leqslant \|\pi(x)\| + \epsilon$, $\|y + n\| \leqslant \|\pi(y)\| + \epsilon$.

$$\begin{aligned} \|\pi(x)\pi(y)\| &= \|\pi(xy)\| = \|\pi((x + m)(y + n))\| \\ &\leqslant \|(x + m)(y + n)\| \\ &\leqslant (\|\pi(x)\| + \epsilon)(\|\pi(y)\| + \epsilon) \end{aligned}$$

Since ϵ is arbitrary $\|\pi(x)\pi(y)\| \leqslant \|\pi(x)\|\|\pi(y)\|$. □

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Proposition 7.5.6. Let $\mathcal{A}$ be a unital Banach algebra. If an element $x \in \mathcal{A}$ is not invertible then x is contained in some maximal ideal.

Proof. $\mathcal{A}x$ is a proper regular ideal. Hence there exists a maximal ideal containing this. □

Proposition 7.5.7. Let $\phi : \mathcal{A} \to \mathbb{C}$ be a nonzero complex homomorphism. Then $\phi^{-1}(0)$ is a regular maximal ideal. $\phi \mapsto \phi^{-1}(0)$ gives a bijection between nonzero complex homomorphisms and regular maximal ideals of $\mathcal{A}$.

Proof. Since $\mathcal{A}/Ker(\phi)$ is isomorphic with a field $ker(\phi)$ is a regular maximal ideal. To show that the correspondence is bijective observe that for a regular maximal ideal $\mathfrak{m}$, $\mathcal{A}/\mathfrak{m}$ is a Banach algebra with every nonzero element being invertible. This is so because otherwise by the above proposition we will get a contradiction to the maximality of $\mathfrak{m}$. Now by the Gelfand-Mazur theorem $\mathcal{A}/\mathfrak{m} \cong \mathbb{C}$. Hence $\mathfrak{m} = ker(\phi)$ where, $\phi : \mathcal{A} \to \mathcal{A}/\mathfrak{m}$ is the quotient map. □

Proposition 7.5.8. Let ω be a nonzero complex homomorphism of $\mathcal{A}$. Then $\|\omega\| \leqslant 1$.

Proof. We have,

$$|\omega(x)| = |\omega(x^n)|^{1/n} \leqslant \|\omega\|^{1/n}\|x^n\|^{1/n}$$

Now taking limit as n goes to infinity we get $|\omega(x)| \leqslant \rho(x) \leqslant \|x\|$. Therefore $\|\omega\| \leqslant 1$. □

Proposition 7.5.9. (i) Let $\Omega(\mathcal{A})$ be the set of all nonzero complex homomorphisms. Then under weak* topology $\Omega(\mathcal{A})$ is a locally compact Hausdorff space.
(ii) If $\mathcal{A}$ is unital, then $\Omega(\mathcal{A})$ is compact.
(iii) For $x \in \mathcal{A}$, $\widehat{x} : \Omega(\mathcal{A}) \to \mathbb{C}$ defined by $\widehat{x}(\omega) = \omega(x)$ gives a homomorphism $\mathcal{F} : \mathcal{A} \to C_0(\Omega(\mathcal{A}))$, called Gelfand transform.
(iv) For $\mathcal{A}$ unital we have $\sigma(x) = \{\widehat{x}(\omega) : \omega \in \Omega(\mathcal{A})\}$. For $\mathcal{A}$ nonunital $\sigma(x) = \{\widehat{x}(\omega) : \omega \in \Omega(\mathcal{A})\} \cup \{0\}$.
(v) $\|\widehat{x}\| = \rho(x)$.

Proof. (i) Let $\Omega' = \Omega \cup \{0\}$ and ω_i be a convergent net in Ω'. Suppose $\omega_i \to \omega$ in weak* topology. Then $\omega(xy) = lim\omega_i(xy) = lim\omega_i(x)\omega_i(y) = \omega(x)\omega(y)$. Therefore ω is a homomorphism. It may be the zero homomorphism. Being a weak* closed subset of the unit ball of $\mathcal{A}^*$ Ω' is compact. Clearly $\{0\}$ is closed. Hence $\Omega(\mathcal{A})$ is locally compact. Suppose $\omega_1 \neq \omega_2 \in \Omega(\mathcal{A})$. Then there exists $x \in \mathcal{A}$ such that $|\omega_1(x) - \omega_2(x)| > \epsilon$ for some $\epsilon > 0$. Note that $\{\omega : |\omega_1(x) - \omega(x)| < \epsilon/3\}$ and $\{\omega : |\omega_2(x) - \omega(x)| < \epsilon/3\}$ are disjoint neighborhoods of ω_1 and ω_2. Hence Ω' is Hausdorff.
(ii) If $\mathcal{A}$ is unital then $\{0\}$ is an isolated point in Ω' because for any other $\omega \in \Omega'\omega(1) = 1$. Hence Ω is compact.
(iii) $\widehat{x} \in C_0(\Omega(\mathcal{A}))$ because for any $\epsilon \geqslant 0, \{\omega : |\widehat{x}(\omega)| \geqslant \epsilon\}$ is compact. Clearly $\mathcal{F}$ is a homomorphism.
(iv) Case 1 $\mathcal{A}$ Unital : If $\lambda \in \sigma(x)$ then $(x - \lambda)$ is not invertible. Hence there exists $\omega \in \Omega(\mathcal{A})$ such that $\omega(x - \lambda) = 0$ or equivalently $\lambda = \widehat{x}(\omega)$. So, $\lambda \in$ Range of $\widehat{x}$. Conversely suppose $\lambda = \widehat{x}(\omega) = \omega(x)$, then $\omega(x - \lambda) = 0$. Hence $\lambda \in \sigma(x)$.
(v) Follows from (iv). Note that this implies that the Gelfand transform is contractive. □

Definition 7.5.10. Let $\mathcal{A}$ be a commutative Banach algebra then $\Omega(\mathcal{A})$ is called the space of characters of $\mathcal{A}$ or the spectrum of $\mathcal{A}$.

7.6 Characters of $L_1(G)$

Let G be a locally compact abelian group and μ, a left invariant Haar measure. Then we have seen the abelian Banach algebra $L_1(G, \mu)$. We wish to identify its space of characters.

Theorem 7.6.1. *Let ω be a character of $L_1(G)$, that is to say that it is a nonzero homomorphism from $L_1(G)$ to the complex numbers. Then there is a continuous homomorphism $\phi : G \to \mathbb{T}$ such that $\omega(f) = \int_G f(g)\phi(g)dg$.*

Proof. In particular ω is a bounded linear functional on $L_1(G)$, hence there exists $\phi \in L_\infty(G)$ such that $\omega(f) = \int_G f(g)\phi(g)dg$.

$$\begin{aligned}\omega(f_1 \star f2) &= \int_G (f_1 \star f_2)(h)\phi(h)dh \\ &= \int_G \int_G f_1(g)f_2(g^{-1}h)\phi(h)dgdh \\ &= \int_G f_1(g)(\int_G L_g(f_2)(h)\phi(h)dh)dg \\ &= \int_G f_1(g)\omega(L_g(f_2))dg\end{aligned}$$

On the other hand

$$\begin{aligned}\omega(f_1 \star f_2) &= \omega(f_1)\omega(f_2) \\ &= \omega(f_2)\int_G f_1(g)\phi(g)dg.\end{aligned}$$

Therefore ,

$$\int_G \omega(f_2)f_1(g)\phi(g)dg = \int_G f_1(g)\omega(L_g(f_2))dg, \forall f_1, f_2 \in L_1(G). \tag{7.10}$$

Since ω is a nonzero homomorphism there exists f_2 such that $\omega(f_2) \neq 0$. It follows from (7.10) that

$$\phi(g) = \frac{\omega(L_g(f_2))}{\omega(f_2)}, a.e \tag{7.11}$$

Note that ϕ is determined upto a set of measure zero. However Part (3) of proposition (**??**) along with (7.11)shows that ϕ is almost everywhere equal to a continuous function namely $\frac{\omega(L_g(f_2))}{\omega(f_2)}$ and we will take this representative. In particular $\phi(e) = 1$. To see that ϕ is multiplicative note that given arbitrary $f_1, f_2 \in L_1(G)$,

$$\begin{aligned}
0 &= \omega(f_1 \star f_2) - \omega(f_1)\omega(f_2) \\
&= \int_G \int_G f_1(g) f_2(g^{-1}h)\phi(h)\,dg\,dh - (\int_G f_1(g)\phi(g)dg)(\int_G f_2(h)\phi(h)dh) \\
&= \int_G \int_G f_1(g) f_2(g^{-1}h)\phi(gg^{-1}h)\,dh\,dg - \int_G \int_G f_1(g) f_2(h)\phi(g)\phi(h)\,dg\,dh \\
&= \int_G \int_G f_1(g) f_2(h')\phi(gh')\,dh'\,dg - \int_G \int_G f_1(g) f_2(h)\phi(g)\phi(h)\,dg\,dh, \\
&\qquad [\text{ substituiting } g^{-1}h = h',] \\
&= \int_G \int_G f_1(g) f_2(h)(\phi(gh) - \phi(g)\phi(h))\,dg\,dh.
\end{aligned}$$

Since ϕ is continuous this shows that ϕ is a homomorphism, that is

$$\phi(gh) = \phi(g)\phi(h), \forall g, h \in G.$$

It remains to show that $|\phi(g)| = 1, \forall g \in G$. Suppose there exists $\alpha > 1$ such that the open set $A_\alpha = \{g \in G : |\phi(g)| > \alpha\}$ is non-empty. Fix a compact subset K of A_α of positive measure. Define

$$f(g) = \begin{cases} \frac{\overline{\phi(g)}}{|\phi(g)|} \text{ if } g \in K, \\ 0, \text{ otherwise} \end{cases} .$$

Then $\|f\|_1 = |K|$, where $|K|$ denotes Haar measure of K. Let $\tilde{f} = \frac{f}{\|f\|_1}$. By proposition (7.5.8) we have

$$\begin{aligned}
1 \geqslant \|\omega\|.\|\tilde{f}\| \geqslant |\omega(\tilde{f})| &= \int_K \frac{\overline{\phi(g)}}{|\phi(g)|}\frac{\phi(g)}{|K|}dg \\
&= \int_K \frac{|\phi(g)|}{|K|}dg > \alpha > 1!
\end{aligned}$$

This contradiction shows that A_α must be empty. That is $|\phi(g)| \leqslant 1$ for all $g \in G$. Similarly considering $\phi(g)^{-1}$ we conclude that $|\phi(g)| \geqslant 1$ for all $g \in G$. Thus we get range of ϕ is contained in $\{z \in \mathbb{C} : |z| = 1\}$.

□

7.7 C^*-algebras

Definition 7.7.1. A Banach algebra A is called involutive if there exists a map $* : A \to A$ such that $a \mapsto a^*$ satisfies

$$\begin{aligned} (a + \lambda b)^* &= a^* + \bar{\lambda} b^*, \\ (ab)^* &= b^* a^*, \\ (a^*)^* &= a, \\ \|x^*\| &= \|x\|. \end{aligned}$$

An involutive Banach algebra A is called a C^*-algebra if $\|x^*x\| = \|x\|^2$ for all $x \in A$.

$x \in A$ is called hermitian or selfadjoint if $x = x^*$, normal if $xx^* = x^*x$, unitary if $x^*x = xx^* = I$, projection if $x = x^* = x^2$.

Proposition 7.7.2. Let A be a C^*-algebra. If $x \in A$ is normal then $\|x\| = \rho(x)$.

Proof. $\|x^2\|^2 = \|(x^2)^*x^2\| = \|(x^*x)^2\| = \|x^*x\|^2 = \|x\|^4$. Therefore we have, $\|x^2\| = \|x\|^2$, implying $\|x^{2^n}\| = \|x\|^{2^n}$. So $\rho(x) = \|x\|$. □

Proposition 7.7.3. Let A be a unital C^*-algebra.

1. $\sigma(u) \subseteq \{\lambda : |\lambda\| = 1\}$ for all unitary u.
2. $\sigma(h) \subseteq \mathbb{R}$ for all hermitian h.

Proof. (1) $\|u\|^2 = \|u^*u\| = \|I\| = 1 \implies \|u\| = 1$. Therefore $\sigma(u)$ is contained in the unit disc. Also u is invertible with $u^{-1} = u^*$. Therefore 0 does not belong to $\sigma(u)$. Therefore by the spectral mapping theorem we have $\sigma(u^{-1}) \subseteq \{z \in \mathbb{C} : |z| \geqslant 1\}$. On the otherhand $\|u^{-1}\| = \|u^*\| = 1$, hence $\sigma(u^{-1}) \subseteq \{z \in \mathbb{C} : |z| \leqslant 1\}$. Therefore $\sigma(u^{-1}) \subseteq \{z \in \mathbb{C} : |z| = 1\}$. Now by the spectral mapping theorem we are done.

(2)$u = e^{ih}$ is a unitary. Hence by the spectral mapping theorem we have $e^{i\sigma(h)} \subseteq \{z \in \mathbb{C} : |z| = 1\}$. The only way this can happen is $\sigma(h) \subseteq \mathbb{R}$. □

Theorem 7.7.4. *Let $\mathcal{A}$ be an abelian C^*-algebra. If Ω is the spectrum of A, then the Gelfand transformation is an isometric isomorphism of A onto $C_0(\Omega)$, preserving the $*$-operation.*

Proof. We know $\|\widehat{x}\| = \rho(x)$. On the other hand since $\mathcal{A}$ is abelian every element is normal. So, $\|x\| = \rho(x)$. Therefore the Gelfand transform $x \mapsto \widehat{x}$ is isometric. Take $\omega \in \Omega$, for $h \in \mathcal{A}_h$, $\omega(h) \in \sigma(h) \subseteq \mathbb{R}$. x can be expressed as $x = h + ik$, with $h, k \in \mathcal{A}_h$. $\omega(x^*) = \omega(h - ik) = \omega(h) - i\omega(k) = \overline{\omega(x)}$. Hence $x \mapsto \widehat{x}$ preserves *-operation.
Let $\mathcal{F} : \mathcal{A} \to C_0(\Omega)$, $\mathcal{F}(x) = \widehat{x}$, then $\mathcal{F}(\mathcal{A})$ separates points because if $\omega_1 \neq \omega_2 \in \Omega$, then there exists $x \in \mathcal{A}$ such that $\omega_1(x) \neq \omega_2(x)$. Hence $\widehat{x}(\omega_1) \neq \widehat{x}(\omega_2)$. By the Stone-Weirstrass theorem$\mathcal{F}\mathcal{A} = C_0(\Omega)$. □

Proposition 7.7.5. Let Ω be a locally compact Hausdorff space and $\mathcal{A} = C_0(\Omega)$. The map $\omega \in \Omega \mapsto \widehat{\omega} \in \Omega(\mathcal{A})$ given by $\widehat{\omega}(x) = x(\omega)$ is a homeomorphism of Ω onto $\Omega(\mathcal{A})$.

Proof. Let us assume Ω to be compact. Then $\Omega(\mathcal{A})$ is compact and $\omega \mapsto \widehat{\omega}$ is continuous because if $\omega_\alpha \to \omega$ then $x(\omega_\alpha) \to x(\omega) \forall x \in \mathcal{A}$, or equivalently $\widehat{\omega}(x) \to \omega(x)$.
$\omega \mapsto \widehat{\omega}$ is one to one: Suppose $\omega_1 \neq \omega_2$, then by Tietze extension theorem $\exists f$ such that $f(\omega_1) = 0$ and $f(\omega_2) = 1$. $\widehat{\omega_1}(f) \neq \widehat{\omega_2}(f)$.
$\omega \mapsto \widehat{\omega}$ is onto: Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{A}$. Then $\exists \omega$ such that $\mathfrak{m} = \{x : x(\omega) = 0\}$. Let ϕ be the homomorphism corresponding to $\mathfrak{m}$,i.e., $\phi : \mathcal{A} \to \mathcal{A}/\mathfrak{m}, \phi(x) = x(\omega)$. Then $\widehat{\omega} = \phi$. So $\omega \mapsto \widehat{\omega}$ is a bijective map between compact Hausdorff spaces. Hence it is a homeomorphism.

If Ω is locally compact and not compact then argue through one point compactification. □

Proposition 7.7.6. Let $\mathcal{B} \subseteq \mathcal{A}$ be a C^*-subalgebra of a unital C^*-algebra containing the identity. Then $\forall x \in \mathcal{B} \sigma_{\mathcal{B}}(x) = \sigma_{\mathcal{A}}(x)$.

Proof. Case 1: Let x be self adjoint.
Clearly $\sigma_{\mathcal{A}}(x) \subseteq \sigma_{\mathcal{B}}(x)$. Suppose $\lambda \in \mathbb{R} \setminus \sigma_{\mathcal{A}}(x)$ we want to show $\lambda \notin \sigma_{\mathcal{B}}(x)$. For $\epsilon > 0, \lambda_\epsilon = \lambda + i\epsilon \notin \sigma_{\mathcal{B}}(x)$, hence $(x - \lambda_\epsilon)^{-1} \in \mathcal{B}$. Using continuity of inverse in $G(\mathcal{A})$, we get $(x - \lambda_\epsilon)^{-1} \to (x - \lambda)^{-1}$ in $G(\mathcal{A})$. Since $\mathcal{B}$ is closed, $(x - \lambda)^{-1} \in \mathcal{B}$, hence $\lambda \notin \sigma_{\mathcal{B}}(x)$.
Case 2: If $x \in \mathcal{B}$ is invertible in $\mathcal{A}$ then x^*x is invertible in $\mathcal{A}$ and so in $\mathcal{B}$ (By

the previous case). Hence x is left invertible in $\mathcal{B}$. Similarly using $xx^*\, x$ is right invertible in $\mathcal{B}$. Hence x is invertible in $\mathcal{B}$. So, $\lambda \notin \sigma_{\mathcal{A}}(x)$ iff $(x-\lambda)$ is invertible in $\mathcal{A}$ iff $(x-\lambda)$ is invertible in $\mathcal{B}$ iff $\lambda \notin \sigma_{\mathcal{B}}(x)$. □

Proposition 7.7.7. Let $\mathcal{A}$ be a unital C^*-algebra. If $x \in \mathcal{A}$ is normal then there exists a unique isomorphism $\phi : C(\sigma(x)) \to C^*(x)$, the C^*-algebra generated by x and 1 such that $\phi(i) = 1, \phi(\iota) = x$ where $\iota : \sigma(x) \to \mathbb{C}$ is the function $\iota(\lambda) = \lambda$.

Proof. Let $\mathcal{B} = C^*(x)$ and $\mathcal{P}$ = polynomials in x and x^*. $\mathcal{P}$ is dense in $\mathcal{B}$. Let Ω = space of all complex homomorphisms from $\mathcal{B}$ to $\mathbb{C}$. Define $\psi : \Omega \to \sigma(x)$ by $\psi(\eta) = \eta(x)$.
$\psi(\eta) \in \sigma(x)$: $\eta(x - \eta(x)) = 0$, hence $x - \eta(x)$ is not invertible.
ψ is continuous: Suppose $\eta_\alpha \to \eta$ in weak*, then $\eta_\alpha(x) \to \eta(x)$ in $\mathbb{C}$.
ψ is one to one: Suppose η_1 and η_2 are two homomorphisms such that $\eta_1(x) = \eta_2(x)$, then $\eta_1|_{\mathcal{P}} = \eta_2|_{\mathcal{P}}$. Since $\mathcal{P}$ is dense in $\mathcal{B}$, $\eta_1 = \eta_2$.
ψ is onto: Suppose $\lambda \in \sigma(x)$, then $\exists \eta$ such that $\lambda = \eta(x)$. $\psi(\eta) = \lambda$.
ψ is a bijective continuous map between compact Hausdorff spaces and hence a homeomorphism. ψ induces an isomorphism between $C(\Omega)$ and $C(\sigma(x))$. This isomorphism composed with the inverse of the Gelfand transform gives the required isomorphism. In other words $\phi(f) = \mathcal{F}^{-1}(f \circ \psi)$ is the isomorphism. □

Definition 7.7.8 (Continuous Function Calculus)**.** Let $x \in \mathcal{A}$ be a normal element. Let f be a complex valued continuous function on $\sigma(x)$. Then $\phi(f)$ with ϕ as in the previous proposition is denoted by $f(x)$.

Proposition 7.7.9. Let $\mathcal{A}$ be a unital C^*-algebra. Then every element of $\mathcal{A}$ is a linear combination of 4 unitary elements.

Proof. Let $x \in \mathcal{A}$ be selfadjoint and $\|x\| \leqslant 1$. $u = x + i(1-x^2)^{1/2}$ is a unitary and $x = \frac{1}{2}(u + u^*)$. □

Proposition 7.7.10. Let $K \subseteq \mathbb{C}$ be compact. $A_K = \{x \in \mathcal{A} | x$ is normal and $\sigma(x) \subseteq K\}$. If $f : K \to \mathbb{C}$ is continuous then $x \in A_L \mapsto f(x) \in \mathcal{A}$ is continuous.

Proof. By Stone-Weirstrass there ixists a polynomial $p(z, \bar{z})$ such that

$$\mathrm{Sup}_{z\in K}|p(z,\bar{z}) - f(z)| < \epsilon$$

There exists a constant M such that $\|x\| < M$ for $x \in A_K$. Also, since p is a polynomial $\exists \delta > 0$ such that

$$\|p(x, x^*) - p(y, y^*)\| < \epsilon \text{ if } \|x - y\| < \delta, \|x\|, \|y\| < M.$$

Now if $x, y \in A_K$ and $\|x - y\| < \delta$, then $\|f(x) - f(y)\| \leqslant \|f(x) - p(x, x^*)\| + \|p(x, x^*) - p(y, y^*)\| + \|f(y) - p(y, y^*\| < 3\epsilon$. □

Theorem 7.7.11. *For a selfadjoint element x in a C^* algebra $\mathcal{A}$, the following are equivalent.*
(i) $\sigma(x) \in [0, \infty)$.
(ii) $x = y^*y$ *for some* $y \in \mathcal{A}$.
(iii) $x = h^2$ *for some* $h \in \mathcal{A}$.
The set of all selfadjoint elements satisfying any of the above is a closed convex cone P *in* $\mathcal{A}$ *with* $P \cap (-P) = \{0\}$

7.8 Assignment-III due on 28/04/25

1. Let $\mathcal{H}$ be a separable Hilbert space. Show that there exists subspaces $\{\mathcal{H}\}_{s\in[0,1]}$ such that for each $0 \leqslant s < t \leqslant 1, \mathcal{H}_s \subsetneq \mathcal{H}_t$.

2. (*) Let E, F be normed linear spaces and $T : E \times F \to \mathbb{K}$ be a bilinear map. Show that T is continuous where $E \times F$ is endowed with the product topology iff $\exists C > 0$ such that $|T(x, y)| \leqslant C\|x\|\|y\|, \forall x \in E, y \in F$.

3. Let $\mathcal{H}, \mathcal{H}'$ be Hilbert spaces. Consider the sesquilinear form on the algebraic tensor product $\mathcal{H} \otimes_{alg} \mathcal{H}'$ given by

$$\langle u \otimes u', v \otimes v' \rangle_{\mathcal{H}\otimes\mathcal{H}'} := \langle u, v\rangle_{\mathcal{H}} \cdot \langle u', v'\rangle_{\mathcal{H}'}.$$

Show that this is a preinner product. Is $\mathcal{H} \otimes_{alg} \mathcal{H}'$ complete with respect to the associated norm? The associated Hilbert space is denoted by $\mathcal{H} \otimes \mathcal{H}'$.

4. (*) Let $(\Omega, \mathfrak{S}, P), (\Omega', \mathfrak{S}', P')$, then show that $L^2(\Omega) \otimes L^2(\Omega')$ is unitarily equivalent with $L^2((\Omega \times \Omega, \mathfrak{S} \otimes \mathfrak{S}', P \otimes P'))$.

5. (*) Let $\|\cdot\|_1, \|\cdot\|_2$ be norms on E such that E is complete with respect to both the norms and there exists $c > 0$ such that $\|x\|_1 \leqslant c\|x\|_2, \forall x \in E$. Then show that both the norms are equivalent.

6. (*) Show that ℓ_1 is not complete in the norm $\|\cdot\|_\infty$.

7. Let E, F be Banach spaces and $T \in \mathcal{L}(E; F)$ be such that $\forall x \in E, \exists n_x$ such that $T^{n_x}(x) = 0$. Then show that there exists n such that $T^n(x) = 0$.

8. (*) For $r > 0$, let $\mathcal{H}_r$ be the Hilbert space given by

$$\mathcal{H}_r = \{\{x_n\}_{n\in\mathbb{Z}} : \forall n \in \mathbb{Z}, x_n \in \mathbb{C}, \|\{x_n\}\|_r := \sqrt{\sum_n (1+n^2)^{r/2}|x_n|^2} < \infty\}.$$

Let $r > s > 0$ and $T : \mathcal{H}_r \hookrightarrow \mathcal{H}_s$ be the inclusion map. Then show that T is a compact operator. (Hint: Show that the image of the unit ball is totally bounded.)

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9. (*) Let $(\Omega, \mathfrak{S}, P)$ be a probability space and $K \in L^2((\Omega \times \Omega, \mathfrak{S} \otimes \mathfrak{S}, P \otimes P))$. Then show that $T_K : L^2(\Omega) \to L^2(\Omega)$ given by

$$(T_K f)(x) := \int K(x,y) f(y) dP(y)$$

is a compact operator.

10. Let $T \in \mathcal{L}(\mathcal{H})$. Then show that
    (a) $\ker(T) = \mathrm{Ran}(T^*)^{\perp}$.
    (b) $\ker(T)^{\perp} = \overline{\mathrm{Ran}(T^*)}$.

11. (*) Let $T \in \mathcal{B}(\mathcal{H})$ be normal. Then show that $\sigma_r(T) = \emptyset$, where $\sigma_r(T)$ is the residual spectrum of T.

12. (*) (Quasi nilpotent) Let $\{\lambda_n\}$ be a sequence of scalars so that $|\lambda_n| \downarrow 0$. Define $T : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ the linear map

$$T(\{x_n\}_{n \in \mathbb{N}} := \{y_n\}_{n \in \mathbb{N}}, \text{ where } y_n = \lambda_n x_{n+1}.$$

Compute $\|T^n\|$ and conclude that $\lim_n \|T^n\|^{1/n} = 0$. Such operators are called quasi nilpotent operator. Let $\mathcal{A} \subseteq \mathcal{B}(\ell_2(\mathbb{N}))$ be the unital Banach algebra generated by T and $\omega : \mathcal{A} \to \mathbb{C}$ be a unital homomorphism. Show that $\omega(T) = 0$.

13. Let E be a vector space over $\mathbb{K}$ and $\phi_j; 1 \leqslant j \leqslant n, \phi$ be linear functionals such that $\cap \ker \phi_j \subseteq \ker \phi$. Show that there exists scalars $\lambda_j; 1 \leqslant j \leqslant n$ such that $\phi = \sum \lambda_j \phi_j$.

14. Let E be a $\mathbb{K}$ vector space and $\mathcal{A}$ be a subspace of the space of linear functionals on E. Note that E is just a vector space and we are considering linear maps. Show that $(E, \sigma(E; \mathcal{A}))^* = \mathcal{A}$. (Hint: use (1c) of assignment II along with the previous problem)

15. Let E, F be Banach spaces and $T : (F^*; \text{weak}^*) \to (E; \text{weak}^*)$ be a continuous linear map. Show that there exists a norm continuous linear map $S : E \to F$ so that $T = S^*$. (Hint: Obtain S by the previous exercise and conclude continuity using closed graph theorem.)

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16. (***) Let $(\Omega, \mathfrak{S})$ be a measurable space and $\xi : \mathfrak{S} \to \mathcal{P}(\mathcal{H})$ be a spectral measure with $\mathcal{H}$ separable. For $u, v \in \mathcal{H}$, let $\xi_{u,v} : \mathfrak{S} \to \mathbb{C}$ be the complex measure $\xi_{u,v}(A) = \langle u, \xi(A)v \rangle$. Let $\mathcal{S}$ be the collection of all simple functions. Show that

$$\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{H}), s \mapsto \sum c_i \xi(A_i)$$

where $s = \sum c_i \chi_{A_i}$ defines a *-homomorphism satisfying the following properties,

(a) $\langle u, \Phi(s)v \rangle = \int s(x) d\xi_{u,v}(x)$.

(b) $\|\Phi(s)v\|^2 = \int |s(x)|^2 d\xi_{v,v}(x)$.

(c) $\|\Phi(s)\| = \inf_{A \in \mathfrak{S}, \xi(A)=0} \sup_{x \in A^c} |s(x)| =: \|s\|_{\infty,\xi}$.

Let $L_\infty(\xi)$ be the completion of $\mathcal{S}$ in the norm $\|\cdot\|_{\infty,\xi}$. Show that Φ extends to a *-homomorphism $\Phi : L_\infty(\xi) \to \mathcal{B}(\mathcal{H})$ satisfying (i)-(iii). We use the notation $\int f d\xi$ to denote $\Phi(f)$.

17. Let $\mathcal{H}$ be a Hilbert space and $\{U_t\}_{t \in \mathbb{R}} \subseteq \mathcal{B}(\mathcal{H})$ be a family of unitaries so that $U_0 = I, U_t \circ U_s = U_{t+s}, \forall t, s \in \mathbb{R}$. Such a family is referred as a one parameter unitary group. Suppose $t \mapsto U_t$ is a continuous map. Then show that $\lim_{t \to 0} \frac{U_t - I}{it}$ exists and is a bounded self-adjoint operator.

18. Let $T\mathcal{B}(\mathcal{H})$ be a normal operator with its spectral measure ξ. Show that $\lambda \in \sigma_p(T)$ iff $\xi(\{\lambda\}) \neq 0$ where $\sigma_p(T)$ denotes the point spectrum of T.