INDIAN STATISTICAL INSTITUTE

Semester	:	Second Semester 2024-25
Course	:	M.Math 1st Year
Subject	:	Functional Analysis
Name	:	
Roll No	:	

Date : 11th February 2025 Marks : Duration: 30 minutes

1. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in a topological space X. For each $\lambda \in \Lambda$, let $F_\lambda \coloneqq \{x_{\lambda'} : \lambda' \succeq \lambda\}$. Let $\mathfrak{B} \coloneqq \{F_\lambda : \lambda \in \Lambda\}$. Let $\mathfrak{F}$ be the filter generated by the filter base $\mathfrak{B}$. Suppose that the net converges to x. Then show that the filter $\mathfrak{F}$ converges to x. **[5]**

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2. Let Λ be an index set and $\mathfrak{U}$ be an ultrafilter on Λ. Suppose we have a function $f : \Lambda \to Y$, where Y is a topological space. We say f has $\mathfrak{U}$-limit y for some y in Y if for all $V \in \mathcal{N}_y, f^{-1}(V) \in \mathfrak{U}$. This is denoted by $\mathfrak{U}\text{-}\lim f = y$. Show that if Y is a compact Hausdorff space and $\mathfrak{U}$ is an ultrafilter on Λ then each $f : \Lambda \to Y$ has a $\mathfrak{U}$-limit. **[5]**

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3. If V is a neighbourhood of origin in a TVS E, then show that for any subset $A \subseteq E$ we have $\overline{A} \subseteq A + V$. **[5]**

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4. Prove that if $A \subseteq E$ is a compact subset of a TVS E and $C \subseteq \mathbb{C}$ is also compact then so is $\cup_{\gamma \in C} \gamma . A \subseteq E$. **[5]**

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5. Let E, F be locally convex spaces with topologies prescribed by families of semi-norms $\mathfrak{P}, \mathfrak{Q}$ respectively. Let $T : E \to F$ be a continuous linear map. Then show that for all $q \in \mathfrak{Q}$ there exists $n \in \mathbb{N}, p_1, \cdots, p_n \in \mathfrak{P}, C > 0$ such that $q(T(x)) \leq C \max_i p_i(x)$. **[5]**

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6. Let $\Omega \subseteq \mathbb{R}^d$ be an open subset. Give an example of a continuous seminorm on $\mathcal{D}(\Omega)$ and establish it's continuity. **[5]**

Solution 1. Set up: Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of locally convex spaces with $E_n \subseteq E_{n+1}, \forall n \in \mathbb{N}$ and for all n the topology of E_n is the subspace topology derived from E_{n+1}. Suppose $E = \cup E_n$ has the strict LF topology.

Relevant result: We know that the collection of seminorms

$$\{p : E \to \mathbb{R} : p \text{ is a seminorm and } p|_{E_n} : E_n \to \mathbb{R} \text{ is continuous } \forall n\}$$

prescribes the topology of E. Therefore to check continuity of a given seminorm$q$ on E it suffices to check continuity of $q|_{E_n}$ for all n.

In the given problem we can take $q(f) = \sup_{x\in\Omega} |f(x)|$. Then $q|_{C^\infty(K)}$ is continuous for all compact K and we are done. □

Solution 2. Relevant result: Let $U_k \subseteq E_k$ be a convex balanced and absorbing open set. Then there exists a sequence of convex, balanced, absorbing open sets $U_n \subseteq E_n, \forall n \geq k$ such that $U_{n+1} \cap E_n = U_n, \forall n \geq k$. Consequently $U = \cup_{n\geq k} U_n \subseteq E$ is a convex balanced and absorbing open set in E. Therefore Minkowski functional p_U of U is a continuous seminorm on E. If $U_k \neq E_k$, then $U \neq E$ and consequently p_U is not identically zero.

In the given problem we take $U_1 = \{f : supp(f) \subseteq K_1, \sup |f(x)| < 1\}$ where $\Omega = \cup K_n$ is a given exhaustion of Ω by compacts. □

Solution 3. Let $\Omega \subseteq \mathbb{R}^d$ be an open subset. Define

$$p : C_c^\infty(\Omega) \ni f \mapsto \sup_{x\in\Omega} |f(x)| \in \mathbb{R}.$$

Obviously p is a seminorm. Therefore $|p(f) - p(g)| \leq p(f-g), \forall f, g \in C_c^\infty(\Omega)$ and to establish continuity it is enough to establish continuity at origin. It suffices to show that the convex, balanced and absorbing set $B_1 \coloneqq \{f : p(f) < 1\}$ is open. That follows once we establish that $B_1 \cap C^\infty(K)$ is open in $C^\infty(K)$ for all compact subset $K \subseteq \Omega$. But that is obvious once we note that

$$B_1 \cap C^\infty(K) = \{f \in C^\infty(K) : \sup_{x \in K} |f(x)| < 1\}.$$

$\square$