ASSIGNMENT 1

MMATH FIRST YEAR, 2025

Problems

1. Show that there is an integer multiple of 7 of the form $111\cdots 1$.
2. Show that $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ is not an integer for any integer $n>1$.
3. Show that for all positive real X sufficiently large, there is a perfect square between X and $X+2\sqrt{X}+1$.
4. (a) If 2^n-1 is a prime, then show that n must be a prime.
(b) A Fermat prime is a prime number of the form $2^{2^n}+1$. Show that a prime of the form 2^n+1 must be a Fermat prime.
5. Show that product of two consecutive positive integers cannot be a perfect square.
6. Suppose α is an irrational number. Show that there infinitely many rational numbers m/n such that

$$\left|\alpha-\frac{m}{n}\right|\leq\frac{1}{n^2}.$$

Hint: Use the pigeonhole principle.
Remark: The above statement is a theorem of Dirichlet and is the first result in the branch of Number Theory called Diophantine Approximation.
7. Determine for what n, n^4+4^n is a prime.
8. Let $f\in\mathbb{Z}[x]$ be a non-constant polynomial. Then show that (a) $f(n)$ is composite for infinitely many $n\in\mathbb{N}$;
and (b) the union of the sets of prime factors of $f(n)$ as n varies over $\mathbb{N}$ is infinite.
9. Suppose $(\ ,\ )$ denotes the *gcd* and $[\ ,\ ]$ denotes the *lcm* of two integers. Show (preferably without using prime factorization) that
(i) $a|bc$ *iff* $\frac{a}{(a,b)}|c$,
(ii) $(a,[b,c])=[(a,b),(a,c)]$,
(iii)$[a,(b,c)]=([a,b],[a,c])$
(iv) $(a,b)=(a+b,[a,b])$.
10. If $ad-bc=1$, show that $(a+b,c+d)=1$.
11. Show that $\binom{2n}{n}$ is divisible by $n+1$.
12. If 2^n+1 is a prime, show that n itself a power of 2.
13. A *perfect number* is a positive integer which equals the sum of all its divisor except itself; e.g., 6=1+2+3. Prove that an even number n is perfect *iff* n is of the form $n=2^{p-1}(2^{p-1}-1)$, where p as well as $2^{p-1}-1$ are primes.
Remark: This is a theorem of Euclid and gives a complete and explicit description of all even perfect numbers. Apparently, the oldest unsolved problem in Mathematics is to determine if there exists an odd perfect number. See the discussion here: https://mathoverflow.net/questions/27075/what-is-the-oldest-open-problem-in-mathematics.