# ASSIGNMENT 4

MMATH FIRST YEAR, 2025

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**Problems**

1. Show that if $g_1$ and $g_2$ are primitive roots for $p$, a prime, then $g_1 g_2$ is not.
2. If $m > 6$ and $m$ has primitive roots, then show that the product of all the primitive roots is congruent to 1 modulo $m$.
3. Find all $n$ such that $2^n \mid (3^n - 1)$. Explain your solution.
4. Let $g$ be a primitive root for $p^n$ for some $n > 1$, where $p$ is a prime. Show that $g$ is also a primitive root for $p$.
5. Suppose $g$ is a primitive root for $n > 2$. What is the smallest positive integer $k$ such that $g^k \equiv -1 \pmod{n}$?
6. Show that for a prime $p$, the sum of the distinct primitive roots for $p$ is congruent to $\mu(p-1)$ modulo $p$.
7. Suppose $f$ is multiplicative. Show that $f$ is completely multiplicative *iff* $f^{-1}(n) = \mu(n)f(n)$ for every $n$.
8. Find all positive integers $n$ such that (i) $\phi(n) = n/2$, (ii) $\phi(n) = 2n$, (iii) $\phi(n) = 12$.
9. Show that a quadratic residue for a prime $p$ cannot be a primitive root for $p$.
10. Given $q > 1$, how many characters are there modulo $q$ and how many of them are real valued? Justify your answer.
11. Suppose $q_1, q_2 > 1$ and $\chi_1$ and $\chi_2$ are two characters of moduli $q_1$ and $q_2$.
(a) Show that $\chi_1\chi_2$ is a character itself of modulus $[q_1, q_2]$.
(b) If $q_1$ and $q_2$ are coprime and $\chi$ is a character of modulus $q = q_1q_2$, then show that $\chi$ can be expressed uniquely as a product of two characters $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$.
12. Describe all the Dirichlet characters modulo 2, 3, 4, 5, 6, 9, 12, 30.

13. Show that
$$\sum_{k|n} d(k)^3 = \left(\sum_{k|n} d(k)\right)^2.$$

14. Let $h$ denote the indicator function for squares of integers. Express $h$ using the Möbius function $\mu$.
15. Show that
$$\sum_{\substack{1\le a\le n\\(a,n)=1}} e^{2\pi i \frac{a}{n}} = \mu(n).$$
16. (i) For which integers $n$, is $d(n)$ odd?
(ii) For which $n$, $\sigma(n)$ is odd?

17. Show that $\phi(n) > n/6$ if $n$ has at most 8 distinct prime factors.
18. Show that

$$\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)}.$$

19. Show that the sum

$$\sum_{d^2|n} \mu(d)$$

is 1 or 0 according as $n$ is square-free or not.
20. Let $\lambda : \mathbb{N} \longrightarrow \mathbb{C}$ be a function that satisfies

$$\lambda(m)\lambda(n) = \sum_{d|(m,n)} \lambda(mn/d^2) \text{ for all integers } m, n \geq 1.$$

Show that for all positive integers $m, n$,

$$\lambda(mn) = \sum_{d|(m,n)} \mu(d)\lambda(m/d)\lambda(n/d).$$

21. Show that for any complex numbers $c_1, c_2, \cdots, c_q$,

$$\sum_{\chi (\text{mod } q)} \left| \sum_{1 \leq n \leq q} c_n \chi(n) \right|^2 = \phi(q) \sum_{\substack{1 \leq n \leq q \\ (n,q)=1}} |c_n|^2.$$

Hint: Expand the square, bring the sum over characters inside and use orthogonality.
22. Let $\chi(\text{mod } p)$ be a non-principal Dirichlet character modulo $p$, a prime. Show that

$$\left| \sum_{a (\text{mod } p)} \chi(a) e^{2\pi i \frac{a}{p}} \right| = \sqrt{p}.$$

Hint: Consider the diagonal and the off-diagonal separately after expanding the square.
23. Show that the sum

$$S_\chi = \sum_{1 \leq n \leq q} n\chi(n)$$

is an integer for any character $\chi(\text{mod } q)$.