ASSIGNMENT 5

MMATH FIRST YEAR, 2025

Problems

1. (Hensel's lemma: original version) Let $f(x) = c_0 + c_1x + \cdots + c_nx^n \in \mathbb{Z}_p[x]$. Suppose $f(x) \equiv 0(\text{mod } p)$ has a solution a_1 (in $\mathbb{Z}_p$) and that $f'(a_1) \not\equiv 0(\text{mod } p)$, where, by f', we mean the formal derivative of f, i.e.,

$$f'(x) = c_1 + 2c_2x \cdots + nc_nx^{n-1}.$$

Show that there is a unique $a \in \mathbb{Z}_p$ such that $a \equiv a_1(\text{mod } p)$ and $f(a) = 0$ in $\mathbb{Z}_p$.
2. Show that $\mathbb{Z}_p$ is uncountable.

3. Find the p-adic expansion of $1/2$ for p odd.

4. Find the 5-adic expansion of $1/3, 1/4, 1/15$ in $\mathbb{Q}_p$

5. Show that the infinite series

$$\sum_{n\geq 1} n!$$

converges in $\mathbb{Q}_p$ with respect to the p-adic norm given by $|\alpha|_p = p^{-v_p(\alpha)}$.
Remark: It is not known if the limit of this sum is in $\mathbb{Q}$ or not!

6. Show that in $\mathbb{Q}_p$

$$\sum_{n\geq 1} n.n! = -1,$$

for any prime p.

7. (Product formula) For $\alpha \in \mathbb{Q}, \alpha \neq 0$, show that

$$\prod_{p\leq\infty} |\alpha|_p = 1,$$

where $|\alpha|_\infty$ denotes the usual absolute value of a real number.
Remark: Hence, the usual absolute value and all the p-adic absolute values should be considered at the same footing.

8. Show that $\mathbb{Z}_p$ is a local ring and find its unique maximal ideal. What are all the ideals of this ring?

9. (a) Show that a sequence $\{a_n\}$ in $\mathbb{Q}_p$ is Cauchy (with respect to the norm $|\cdot|_p$) *iff* the sequence of real numbers $\{|a_{n+1} - a_n|_p\}$ converges to zero.
Hint: Write $a_{n+k} - a_n$ as a telescoping sum.

(b) Show that for a sequence $\{a_n\}$ in $\mathbb{Q}_p$, the series

$$\sum_{n\geq 1} a_n$$

converges in $\mathbb{Q}_p$ if the series

$$\sum_{n\geq 1} |a_n|_p$$

converges in $\mathbb{R}$

(c) Show that for a sequence $\{a_n\}$ in $\mathbb{Q}_p$, the series

$$\sum_{n\geq 1} a_n$$

converges in $\mathbb{Q}_p$ *iff* $a_n \to 0$ in the p-adic norm. Furthermore, for such a convergent series, show that

$$\left|\sum_{n\geq 1} a_n\right|_p \leq \max_n |a_n|_p.$$

10. Suppose $a_n \to a$ in $\mathbb{Q}_p$. Show that either $|a_n|_p \to 0$ or there is some positive integer N such that $|a_n|_p = |a|_p$ for all $n > N$.

11. (a) Show that $\mathbb{Q}_p$ and $\mathbb{R}$ are not isomorphic as fields.
(b) Show that if p and q are distinct primes then $\mathbb{Q}_p$ and $\mathbb{Q}_q$ are not isomorphic as fields.

12. Show that $\mathbb{Q}_p$ is isomorphic to $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ as rings.

13. Recall that we can represent any $\alpha \in \mathbb{Q}_p$ as a finite tailed Laurent series

$$\sum_{n>-N} a_n p^n,$$

for some $N \geq 0$ and $a_n \in \{0, 1, \cdots, p-1\}$. Show the map

$$\phi : \sum_{n>-N} a_n p^n \to \sum_{n>-N} a_n p^{-n}$$

is a continuous map from $\mathbb{Q}_p$ to $\mathbb{R}$ and determine the image of ϕ.

14. Show that (a) the sphere $S(a,r) = \{x \in \mathbb{Q}_p : |x-a|_p = r\}$, where $a \in \mathbb{Q}_p$ is fixed and $r > 0$ is an open subset of $\mathbb{Q}_p$ (w.r.t. the p-adic metric topology);
(b) the ball $B(a,r) = \{x \in \mathbb{Q}_p : |x-a|_p < r\}$ both open and closed;
(c) if $b \in B(a,r)$ then $B(a,r) = B(b,r)$,.i.e., every point inside a ball is its centre;
(d) two balls have either empty intersection or one is contained in another;
(e) $\mathbb{Q}_p$ is locally compact (Hint: relate a ball with $\mathbb{Z}_p$);
(f) $\mathbb{Q}_p$ is *totally disconnected*, i.e., any nonempty open subset is a singleton.

15. Show that the set of natural numbers is dense in $\mathbb{Z}_p$.

16. Suppose $p \neq 2$ is a prime. (a) Show that $a \in \mathbb{Z}$ has a square root in $\mathbb{Q}_p$ *iff* a is a quadratic residue modulo p.
(b) Show that a p-adic unit $a_0 + pa_1 + p^2a_2 + \cdots \in \mathbb{Z}_p$ is a square in $\mathbb{Z}_p$ *iff* a_0 is a

quadratic residue modulo p.
(c) Show that if $p \neq 2$, ${\mathbb{Q}_p}^\times/({\mathbb{Q}_p}^\times)^2$ has order 4 and find a set of representatives of the cosets.