ASSIGNMENT 6

MMATH FIRST YEAR, 2025

Problems

1. (a) Determine
$$\alpha_0 := \inf\{\alpha > 0 : \int_2^\infty \frac{1}{x(\log x)^\alpha} \mathrm{d}x < \infty\}.$$
(b) Determine
$$\beta_0 := \inf\{\beta > 0 : \sum_{n\geq 1} \frac{1}{n(\log n)^\beta} < \infty\}.$$

2. Find asymptotic formulae for the following sums. You may use results derived in the class.

(a) $\sum_{n\leq x} \frac{\log n}{n}$, (b) $\sum_{n\leq x} \frac{1}{n^{2/3}}$, (c) $\sum_{n\leq x} \frac{\log n}{\sqrt{n}}$, (d) $\sum_{n\leq x} \frac{n}{e^n}$, (e) $\sum_{n\leq x} \frac{1+\cos n}{\sqrt{n}}$, (f) $\sum_{n\leq x} \frac{1}{n^2 \log n}$, (g) $\sum_{n\leq x} \log(x/n)$, (h) $\sum_{n\leq x} \{x/n\}$ (the fractional part).

3. Show that $\Lambda(n) = -\sum_{d|n} \mu(d) \log d$.

4. Show that $\sum_{d|n} \mu(d)(\log \frac{n}{d})^k = 0$ if n has more than k distinct prime factors.

5. Show that $\sum_{n\leq x} \log n = \sum_{n\leq x} [x/n]\Lambda(n)$.

6. Find asymptotic formula for the following sums (p denotes primes):
(a) $\sum_{n\leq x} \frac{d(n)}{n}$, (b) $\sum_{n\leq x} \phi(n) \log n$, (c) $\sum_{n\leq x} \frac{\phi(n)}{n}$, (d) $\sum_{n\leq x} \frac{1}{\phi(n)}$, (e) $\sum_{n\leq x} \frac{1}{p \log p}$, (f) $\sum_{n\leq x} \omega(n)$, where $\omega(n)$ denotes the number of distinct prime factors of n, (g) $\sum_{n\leq x} \mu^2(n)$, (h) $\sum_{n\leq x} \sigma(n)$, (i) $\sum_{n\leq x} \frac{\sigma(n)}{n}$, (j) $\sum_{n\leq x} d^2(n)$.