ASSIGNMENT 7

MMATH FIRST YEAR, 2025

The goal of this assignment is to recall the proof of Dirichlet's theorem on primes in arithmetic progressions

Problems

1. (a) Show that

$$\sum_{n\le y}\frac{1}{\sqrt{n}} = 2\sqrt{y} + c + O(1/\sqrt{y}),$$

where $c > 0$ is an explicit constant.
(b) Show that if $\chi (\text{mod } q)$ is a non-trivial character, then

$$\sum_{n\le y}\frac{\chi(n)}{\sqrt{n}} = L(1/2,\chi) + O(1/\sqrt{y}),$$

2. Define

$$\lambda_\chi(n) = \sum_{d|n}\chi(d)$$

and

$$S(X) = \sum_{n\le X}\frac{\lambda_\chi(n)}{\sqrt{n}}.$$

Using the above two asymptotic formulae and applying Dirichlet's hyperbola method, show that if $\chi \neq \chi_0$, then

$$S(X) = 2\sqrt{X}L(1,\chi) + O(1).$$

3. If χ is a real character, show that $\lambda_\chi(m^2) \geq 1$ for any integer $m > 0$. Conclude that

$$S(X) \longrightarrow \infty \text{ as } X \to \infty.$$

4. Using the above two problems, show that if χ is a non-trivial real character, then $L(1,\chi) \neq 0$.

5. (a) Show that for $\Re s > 1$,

$$\sum_{\chi (\text{mod } q)} \log L(s,\chi) > 0.$$

(b) Using (a) show that $L(1,\chi) \neq 0$ for any Dirichlet character taking non-real values.

6. Show that if $q > 1$, $(a,q) = 1$ and $\Re s > 1$, then

$$\frac{1}{\phi(q)}\sum_{\chi (\text{mod } q)} \overline{\chi}(a)\log L(s,\chi) = \sum_{p\equiv a (\text{mod } q)}\frac{1}{p^s} + O(1),$$

where $O(1)$ is bounded by an absolute constant, i.e., it does not depend on s.

7. Using Problem 4 and 5(b) and properties of the Riemann zeta function, show that

$$\frac{1}{\phi(q)} \sum_{\chi (\mathrm{mod}\ q)} \overline{\chi}(a) \log L(\sigma, \chi) \longrightarrow \infty$$

as $\sigma \to 1+$. Conclude Dirichlet's theorem using Problem 6.