Course Name: M. Math, 1^{st} year
Subject Name : Topology II
Assignment 1

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1. Find a sΔ-set Y such that $|Y| \cong \mathbb{R}P^2$ and compute $H_*^\Delta(Y)$.
2. Repeat the above problem for the torus, and the Klein bottle. [Hint : Interpret the diagrams in Hatcher page 102 as sΔ-sets]
3. Prove that for any sΔ-set Y, $H_0^\Delta(Y)$ is isomorphic to the free Abelian group on the set of path components of $|Y|$.
4. Let $\mathbf{n}$ be the set $\{0, 1, \cdots, n\}$. For $0 \le i \le n$ Consider the abstract simplicial complex $(V, \mathcal{F}_i)$ defined by
$$V = \mathbf{n}, \mathcal{F}_i = \mathcal{P}(\mathbf{n}) \setminus \{\mathbf{n}, \mathbf{n} \setminus \{i\}\}.$$
Show that the geometric realization $|(V, \mathcal{F}_i)|$ is contractible.
5. Let Y be the $s\Delta$-set given by
$$Y_0 = \{v_0, v_1, v_2\},\ Y_1 = \{a, b, c\},\ Y_2 = \{\Delta_1, \Delta_2\},\ Y_n = \varnothing \text{ if } n \ge 3,$$
$$d_0(a) = d_1(b) = v_1,\ d_0(b) = d_0(c) = v_2,\ d_1(c) = d_1(a) = v_0,$$
$$d_0(\Delta_1) = d_0(\Delta_2) = b,\ d_1(\Delta_1) = d_1(\Delta_2) = c,\ d_2(\Delta_1) = d_2(\Delta_2) = a.$$
a) Prove that $|Y| \cong S^2$.
b) Calculate the homology groups $H_*^\Delta(Y)$.
6. a) For a partially ordered set P with partial order $\le$, define $Y_q(P)$ = the set of q-chains, that is, tuples $y_0 \le y_1 \cdots \le y_q$. Define $d_i : Y_q(P) \to Y_{q-1}(P)$ to be the function which removes y_i from the above tuple. Prove that this defines a sΔ-set $Y(P)$.
b) Let $\underline{n}$ be the partially ordered set $0 \le 1 \le \cdots \le n$. Prove that for any sΔ-set Z,
$$\text{Map}_{s\Delta\text{-set}}(Y(\underline{n}), Z) \cong Z_n.$$
c) Prove that $|Y(\underline{n})| \cong \Delta^n$.
d) Compute $H_*^\Delta(Y(\underline{n}))$.
7. Let $Y(\underline{n})$ be as in the above exercise. Define $Y'[n]$ as the sΔ-set defined by
$$Y'[n]_q = \begin{cases} Y(\underline{n})_q & \text{if } q < n \\ \varnothing & \text{if } q \ge n. \end{cases}$$
a) Prove that $|Y'[n]| \cong S^{n-1}$.
b) Compute $H_*^\Delta Y'[n]$.
8. Let $Y'[n]$ be as in the previous exercise. Define $\Lambda_i[n]$ as the sΔ-set
$$\Lambda_i[n]_q = \begin{cases} Y'[n]_q & \text{if } q \le n-2 \\ Y'[n]_{n-1} - \sigma_i & \text{if } q = n-1 \\ \varnothing & \text{if } q \ge n. \end{cases}$$
where $\sigma_i = d_i(0 \le 1 \le \cdots \le n)$ in the notation of exercise 5.
a) Prove that $|\Lambda_i[n]| \simeq *$.
b) Compute $H_*^\Delta(\Lambda_i[n])$.

9. Consider the sequence of Abelian groups C_n with $C_n = 0$ if $n \geq 3$, $C_2 = \mathbb{Z}^2$, $C_1 = \mathbb{Z}^4$, $C_0 = \mathbb{Z}^2$, together with maps between them as in the diagram below

$$\cdots 0 \to 0 \to \cdots 0 \to \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} -3 & 3 \\ 1 & 1 \\ 1 & -2 \\ -2 & 1 \end{bmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{bmatrix} 1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 2 \end{bmatrix}} \mathbb{Z}^2$$

a) Prove that C is a chain complex.
b) Compute $H_*(C)$

10. Let $\psi : C_* \to D_*$ be a chain map. Let C_ψ (the mapping cone of ψ) be defined by

$$(C_\psi)_n = C_{n-1} \oplus D_n, d_n^{C_\psi}(a, b) = (-d_{n-1}^C(c), \psi(c) + d_n^D(c')).$$

1) Prove that C_ψ is a chain complex.
2) Let $S(C)$ be defined as $S(C)_n = C_{n-1}$ with evident differentials. Prove that the projection onto the first factor gives a chain map $C_\psi \to S(C)$.
3) Prove that $0 \to D \to C_\psi \to S(C) \to 0$ is a short exact sequence of chain complexes.
4) Let $Co(C)$ be the mapping cone of the identity map from C to C. Show that ψ is chain homotopic to the 0 map if and only if ψ extends to a chain map from $Co(C) \to D$.

11. Let $A_\bullet = \{A_n \mid n \geq 0\}$ be a sequence of finitely generated Abelian groups. Prove that there is a chain complex $C_\bullet$ such that each C_n is free Abelian and for all $n \geq 0$, $H_n(C_\bullet) = A_n$.

12. Prove that chain homotopy of chain maps between chain complexes is an equivalence relation.

13. Let $C_\bullet$ be a chain complex where each $C_n \cong \mathbb{Q}^{r_n}$ (for some $r_n \in \mathbb{Z}_{\geq 0}$) for all $n \geq 0$. Assume that $H_n(C_\bullet) = 0$ for all $n \geq 0$. Prove that the identity map of $C_\bullet$ is chain homotopic to the 0-map.

14. Let $C_\bullet$ be a chain complex such that for every n, C_n is finitely generated and free Abelian. Prove that for each n, $\text{Ker}(d_n)$ is a summand of C_n. (Def : A summand H of an Abelian group K is a subgroup for which $K = H \oplus A'$)

15. Let $C_\bullet$ be a chain complex such that for every n, C_n is finitely generated and free Abelian. Let $D_\bullet$ be another chain complex. Let $\phi_\bullet = \{\phi_n \mid \phi_n \in \text{Hom}(H_n(C_\bullet), H_n(D_\bullet)\}$ be a sequence of homomorphisms. Prove that there is a chain map $\tilde{\phi} : C_\bullet \to D_\bullet$ such that $\tilde{\phi}_* : H_n(C_\bullet) \to H_n(D_\bullet)$ equals ϕ_n.