Course Name: M. Math, 1^{st} year
Subject Name : Topology II
Assignment 2

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1. Find the homology $H_*(S^n, \{p_1, \cdots, p_k\})$ where $p_1, \cdots, p_k$ are k distinct points on S^n.
2. a) For a point p in S^n find $H_*(S^n, S^n - p)$.
b) Do the same exercise for the Torus, and complex projective space.
3. Let T be the torus which is a CW complex whose 1-skeleton is $S^1 \vee S^1$ (denoted by a and b). Compute $H_*(T/S^1_a)$.
4. Prove that the sphere S^n does not retract to it's equator.
5. Find a space X with $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}$, $H_2(X) = \mathbb{Z}/2$ and $H_i(X) = 0$ if $i > 2$.
6. Find the homology groups of the Klein bottle.
7. Let X be a space and p be the cone point of $C(X)$. Compute $H_*(C(X), C(X) - p)$.
8. Let C be the circle on the torus T which is the image of the line $px = qy$ under the covering map $\mathbb{R}^2 \to T$ [this identifies $T = S^1 \times S^1$ and sends (a, b) to their image modulo $\mathbb{Z} \times \mathbb{Z}$]. Let $X = T/C$ the quotient space obtained by identifying C to a point. Calculate $H_*(X)$.
9. Calculate $H_*(S^3 \times S^5)$, $H_*(S^3 \vee S^5)$, and the map induced by the inclusion $S^3 \vee S^5 \to S^3 \times S^5$.
10. Calculate $H_*(M(n))$ where $M(n)$ is obtained by attaching a 2-cell to S^1 via a map of degree n.
11. a) Suppose X is an n-dimensional CW complex with exactly one n-cell. We say that the top cell e^n splits off from X if the attaching map $\partial e^n \to X^{(n-1)}$ is null homotopic. Show that in this case $X \simeq X^{(n-1)} \vee S^n$.
b) Consider the CW complex structure on $\mathbb{R}P^{2n}$ with one i-cell for every $0 \leq i \leq 2n$. Show that the top cell does not split off from $\mathbb{R}P^{2n}$.
12. Let $M(\mathbb{Z}/n, k) = C(f_n : S^k \to S^k)$ where f_n is a map of degree n.
a) Describe a CW complex structure on $M(\mathbb{Z}/2, 5) \times M(\mathbb{Z}/3, 7)$.
b) Compute $H_*(M(\mathbb{Z}/2, 5) \times M(\mathbb{Z}/3, 7))$.
13. Write down a map f from the torus T to itself such that the map $f_* : H_1(T) \to H_1(T)$ is given by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
14. Consider the map $\phi : \mathbb{R}P^2 \to \mathbb{R}P^2$ given by $\phi([x_0, x_1, x_2]) = [x_0^3, x_1^3, x_2^3]$. Compute the action of ϕ_* on homology groups.
15. Let $M(\mathbb{Z}/n, k) = C(f_n)$ where $f_n : S^k \to S^k$ is a map of degree n. ($C(f)$ denotes the cone of f)
a) Compute $H_*(M(\mathbb{Z}/2, 5); \mathbb{Z})$ and $H_*(M(\mathbb{Z}/2, 5); \mathbb{Z}/2)$.
b) Consider the composite $S^5 \xrightarrow{f} S^5 \xrightarrow{g} S^5$ where f, g are maps of degree 2. Show that we obtain a map ψ as the composite
$$M(\mathbb{Z}/4, 5) \simeq C(g \circ f) \to C(g) \simeq M(\mathbb{Z}/2, 5)$$
(i.e, verify the existence of the arrows and the homotopy equivalences)
d) Compute the effect of ψ_* on homology groups.
16. Compute $H_*(\mathbb{R}P^n, \mathbb{R}P^n - \mathbb{R}P^i; \mathbb{Z}/2)$.